decreases from $\frac{3}{2}$ to 0. Now as θ increases from $2\pi/3$ to π, $\cos\theta$ decreases from $-\frac{1}{2}$ to -1, which means that r decreases from 0 to $-\frac{1}{2}$. We conclude that for θ in $(2\pi/3, \pi)$, r is negative, so that the graph lies in the fourth quadrant instead of the second quadrant (Figure 7.71). Thanks to symmetry, we can deduce the shape of the rest of the limaçon. Does its appearance justify the name limaçon, which means "snail"? □

EXERCISES 7.7

1. Find the Cartesian coordinates of the following points, which are in polar coordinates.
   a. $(3, \pi/4)$
   b. $(-2, -\pi/6)$
   c. $(3, 7\pi/3)$
   d. $(5, 0)$
   e. $(-2, \pi/2)$
   f. $(-2, 3\pi/2)$
   g. $(4, 3\pi/4)$
   h. $(0, 6\pi/7)$
   i. $(-1, 23\pi/3)$
   j. $(-1, -23\pi/3)$
   k. $(1, 3\pi/2)$
   l. $(3, -5\pi/6)$
2. Find all sets of polar coordinates for each of the following points, which are in Cartesian coordinates.
   a. $(3, 3)$
   b. $(4, -4)$
   c. $(0, 5)$
   d. $(-4, 0)$
   e. $(3, 3\sqrt{3})$
   f. $(-\frac{1}{3}, \sqrt{3}/3)$
   g. $(-3, \sqrt{3})$
   h. $(-2\sqrt{3}, 2)$
   i. $(0, 0)$
   j. $(-5\sqrt{3}, -5)$

In Exercises 3–11, write each equation in polar coordinates. Express the answer in the form $r = f(\theta)$ wherever possible.

3. $2x + 3y = 4$
4. $y^2 = 4x$
5. $x^2 + 9y^2 = 1$
6. $9x^2 + y^2 = 4y$
7. $(x^2 + y^2)^2 = x^2 - y^2$
8. $x^2 + y^2 = 2x(3y^2 - x^2)$
9. $x^2 + y^2 = x(x^2 - 3y^2)$
10. $y^2 = \dfrac{x^3}{2 - x}$
11. $y^2 = \dfrac{x^2(3 - x)}{1 + x}$

In Exercises 12–18 write each polar equation as an equation in Cartesian coordinates.

12. $r = 5$
13. $r = 3\cos\theta$
14. $\tan\theta = 6$
15. $\cot\theta = 3$
16. $r\cot\theta = 3$
17. $r = \sin 2\theta$
18. $r = 2\sin\theta\tan\theta$

In Exercises 19–37 sketch the graph of each equation. Each graph has a familiar form.

19. $r = 5$
20. $r = -2$
21. $r = 0$
22. $r = \sin\theta$
23. $r = -\frac{3}{2}\cos\theta$
24. $3 = r\sin(\theta - \pi/2)$
25. $-2 = r\cos(\theta + \pi/4)$
26. $\theta = 3\pi/2$
27. $\theta = -7\pi/6$
28. $|\theta| = \pi/3$
29. $r\sin\theta = 5$
30. $r\cos(\theta - \pi/3) = 2$
31. $r = 2\cot\theta\csc\theta$
32. $r = -3\tan\theta\sec\theta$
33. $r(\sin\theta + \cos\theta) = 1$
34. $r^2(4\cos^2\theta + \sin^2\theta) = 4$
35. $r^2(\cos^2\theta - 9\sin^2\theta) = 9$
36. $r = \dfrac{2}{3\cos\theta - 2\sin\theta}$
37. $r = \dfrac{-1}{\cos\theta + 4\sin\theta}$

In Exercises 38–57 sketch the graph of the given equation. Note any symmetries.

38. $r = 3\sin 2\theta$ (four-leaved rose)
39. $r = 2\cos 2\theta$ (four-leaved rose)
40. $r = -4\cos 3\theta$ (three-leaved rose)
41. $r = -4\sin 3\theta$ (three-leaved rose)
42. $r = -\sin 4\theta$ (eight-leaved rose)
43. $r = 2\cos 6\theta$ (twelve-leaved rose)
44. $r = \cos\dfrac{\theta}{2}$ (*Hint:* Not all symmetry can be deduced from Table 7.1.)
45. $r = \sin\dfrac{\theta}{2}$ (*Hint:* Not all symmetry can be deduced from Table 7.1.)
46. $r^2 = \sin\theta$
47. $r^2 = 25\cos\theta$
48. $r^2 = 9\sin 2\theta$ (lemniscate of Bernoulli)
49. $r^2 = 4\cos 2\theta$ (lemniscate of Bernoulli)

50. $r = 1 + 2 \sin \theta$ (limaçon of Pascal)

51. $r = 2 - \cos \theta$ (limaçon of Pascal)

52. $r = 3(1 - \sin \theta)$ (cardioid)

53. $r = 3[1 - \cos(\theta - \pi/2)]$ (cardioid)

54. $r = 3 \tan \theta$ (kappa curve)
(*Hint:* Find $\lim_{\theta \to \pi/2^-} r \cos \theta$.)

55. $r\theta = 2$ (hyperbolic spiral)

56. $r = 2\theta$ (spiral of Archimedes)

57. $r = \sin \theta + \cos \theta$ (*Hint:* Transform the equation into Cartesian coordinates first.)

58. Show that each of the following pairs of equations has the same graph.
a. $r = 3(\cos \theta + 1)$ and $r = 3(\cos \theta - 1)$ (*Hint:* Show that if (r, θ) satisfies one equation, then $(-r, \theta + \pi)$ satisfies the other.)
b. $r = 2(\sin \theta + 1)$ and $r = 2(\sin \theta - 1)$
c. $r = \theta$ and $r = \theta - 2\pi$

59. Show that the graphs of the equations $r = \frac{1}{2}(1 + \cos \theta)$ and $r^2 = -\cos \theta$ intersect at $(1, 0)$, even though no single set of polar coordinates for $(1, 0)$ satisfies both equations.

*60. Find a polar equation of the collection of points the product of whose distances from the points $(1, 0)$ and $(-1, 0)$ is 1.

7.8 AREA IN POLAR COORDINATES

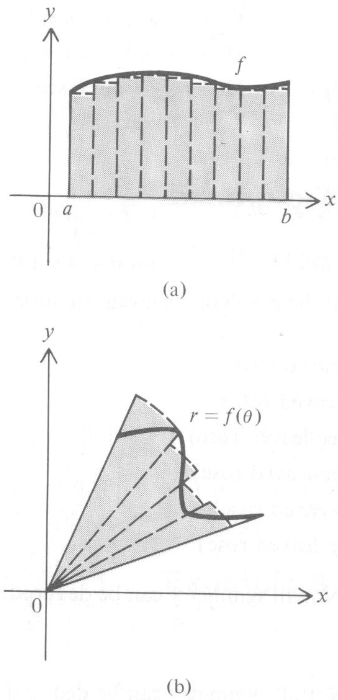

FIGURE 7.72

As we saw in Section 7.7, many regions such as circles and cardioids are easier to describe in polar coordinates than in rectangular coordinates. In most cases it is also easier to compute the areas of such regions using polar coordinates.

When we set out to find the area of a region in rectangular coordinates (in Chapter 5), we divided the region into vertical strips, approximated the area of each such strip by the area of a rectangle, and added the estimates (Figure 7.72(a)). To find the area of a region in polar coordinates we will divide the region into "pie-shaped" sectors (Figure 7.72(b)), approximate the areas of the sectors, and find their sum.

We begin by determining the area of the sector S of a circle with radius r whose sides are on the lines $\theta = 0$ and $\theta = \alpha$, where $\alpha \geq 0$ (Figure 7.73(a)).

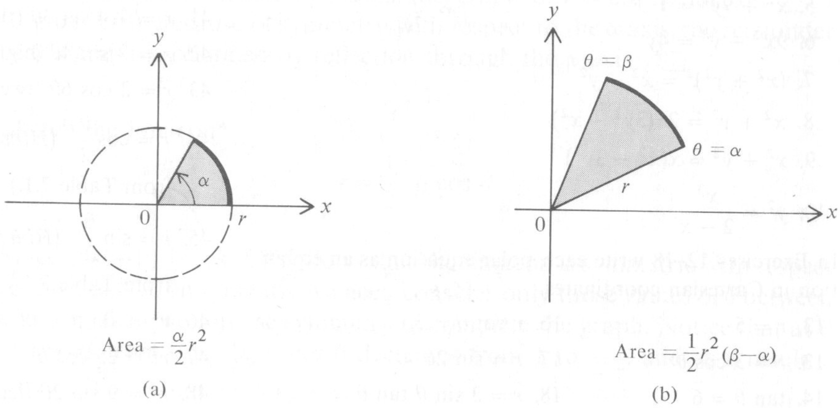

FIGURE 7.73

EXERCISES 7.8

In Exercises 1–16 find the area of the region bounded by the graphs of the given equations.

1. $r = 4$
2. $r = a$, where $a > 0$
3. $r = 3 \sin \theta$
4. $r = 3 \sin \theta$ for $0 \leq \theta \leq \pi/3$, and the line $\theta = \pi/3$
5. $r = -2 \cos \theta$
6. $r = 9 \sin 2\theta$ (four-leaved rose)
7. $r = 9 \cos 2\theta$ for $\pi/4 \leq \theta \leq \pi/2$, and the line $\theta = \pi/2$
8. $r = -4 \sin 3\theta$ (three-leaved rose)
9. $r = \frac{1}{2} \cos 3\theta$ (three-leaved rose)
10. $r = 6 \sin 4\theta$ (eight-leaved rose)
11. $r = 2(1 - \sin \theta)$ (cardioid)
12. $r = 2 + 2 \cos \theta$ (cardioid)
13. $r = 4 + 3 \cos \theta$ (limaçon)
14. $r^2 = 9 \sin 2\theta$ (lemniscate)
15. $r^2 = 25 \cos \theta$
16. $r^2 = -\cos \theta$

In Exercises 17–23 find the area of the region inside the first curve and outside the second curve.

17. $r = 5$ and $r = 1$
18. $r = 5$ and $r = 2(1 + \cos \theta)$
19. $r = 1$ and $r = \sin \theta$
20. $r = 1$ and $r = \cos 2\theta$
21. $r = 1$ and $r^2 = \cos 2\theta$
22. $r = 5(1 + \cos \theta)$ and $r = 2 \cos \theta$
23. $r = 2 + \cos \theta$ and $r = -\cos \theta$

In Exercises 24–28 find the area of the indicated region.

24. The region common to the two circles $r = \cos \theta$ and $r = \sin \theta$.
25. The region common to the circle $r = \cos \theta$ and the cardioid $r = 1 - \cos \theta$.
26. The region inside the circle $r = \cos \theta$ and outside the cardioid $r = 1 - \cos \theta$.
27. The region outside the cardioid $r = 1 + \cos \theta$ and inside the cardioid $r = 1 + \sin \theta$.

*28. The region outside the small loop and inside the large loop of $r = 1 + 2 \cos \theta$.

29. The graph of

$$y^2 = \frac{x^2(1 + x)}{1 - x}$$

is called a *strophoid* (Figure 7.79).

a. Show that $r = \sec \theta - 2 \cos \theta$ is a polar equation of this strophoid.

b. Find the area enclosed by the loop of the strophoid.

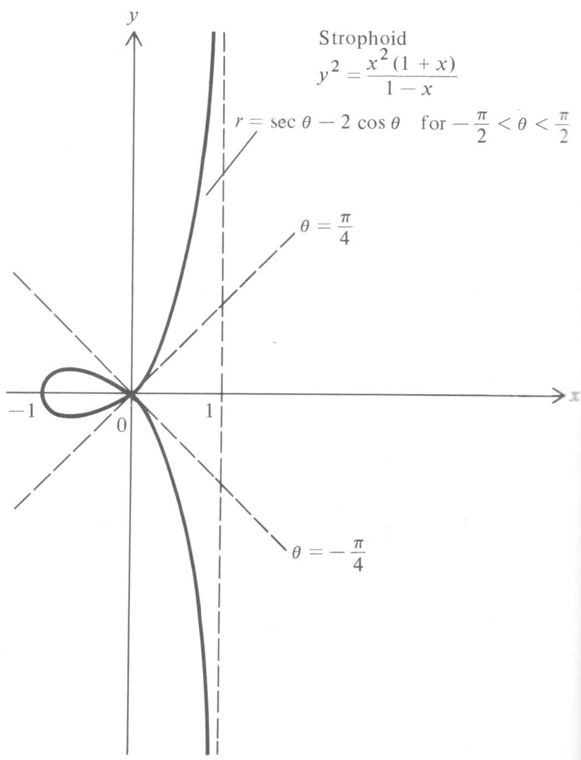

FIGURE 7.79