Cálculo B - Prova 1

Nome:

1. $\displaystyle\int \frac{\ln^3 x}{x\sqrt{\ln^2 x - 4}}\, dx$ [1.5]

2. $\displaystyle\int \frac{dx}{1+\sin x - \cos x}$ [1.5]

3. $\displaystyle\int \sin^2 2t\ \cos^4 2t\ dt$ [2.5]

4. $\displaystyle\int \frac{dx}{1+\sec\frac{x}{2}}$ [2.0]

5. $\displaystyle\int \frac{2x^2 - x + 2}{x^5 + 2x^3 + x}\, dx$ [2.5]

1. $\displaystyle\int \frac{\ln^3 x}{x\sqrt{\ln^2 x - 4}}\, dx$

$u = \ln x \quad , \quad du = \frac{1}{x}$

$\therefore \displaystyle\int \frac{\ln^3 x}{x\sqrt{\ln^2 x - 4}}\, dx = \int \frac{u^3\, du}{\sqrt{u^2-4}}$

$u = 2\sec\theta \quad , \quad du = 2\sec\theta\, tg\theta\, d\theta$

$\therefore \displaystyle\int \frac{u^3\, du}{\sqrt{u^2-4}} = \int \frac{8\sec^3\theta\; 2\sec\theta\, tg\theta\, d\theta}{\sqrt{4\sec^2\theta - 4}}$

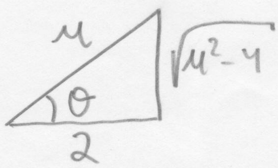
$\displaystyle = 8\int \frac{\sec^4\theta\, tg\theta}{\sqrt{tg^2\theta}}\, d\theta$

$\displaystyle = 8\int \sec^4\theta\, d\theta$ [0.5]

$\displaystyle = 8\int \sec^2\theta\,(1+tg^2\theta)\, d\theta$

$\displaystyle = 8\int \sec^2\theta\, d\theta + 8\int \sec^2\theta\, tg^2\theta\, d\theta$

$$= 8\,tg\,\theta + \frac{8\,tg^3\theta}{3}$$ [1.0]



$$tg\,\theta = \frac{\sqrt{u^2-4}}{2}$$

$$= 8\,\frac{\sqrt{u^2-4}}{2} + \frac{8}{3}\left(\frac{\sqrt{u^2-4}}{2}\right)^3$$

$$= 4\sqrt{u^2-4} + \frac{1}{3}\left(u^2-4\right)^{3/2}$$

$$= 4\sqrt{\ln^2 x-4} + \frac{1}{3}\left(\ln^2 x-4\right)^{3/2}$$ [1.5]

$$\boxed{\int \frac{\ln^3 x}{x\sqrt{\ln^2 x-4}}\,dx = 4\sqrt{\ln^2 x-4} + \frac{1}{3}\left(\ln^2 x-4\right)^{3/2} + C}$$

2. $$\int \frac{dx}{1+\sin x - \cos x}$$

$$z = \operatorname{tg}\frac{x}{2}, \quad \begin{cases} \sin x = \dfrac{2z}{1+z^2} \\ \cos x = \dfrac{1-z^2}{1+z^2} \\ dx = \dfrac{2\,dz}{1+z^2} \end{cases}$$

$$\therefore \int \frac{dx}{1+\sin x - \cos x} = \int \frac{\frac{2\,dz}{1+z^2}}{1+\frac{2z}{1+z^2}-\frac{1-z^2}{1+z^2}} =$$

$$= \int \frac{\frac{2\,dz}{1+z^2}}{\frac{1+z^2+2z-1+z^2}{1+z^2}} = \int \frac{2\,dz}{2z^2+2z}$$

$$= \int \frac{dz}{z(z+1)}$$ [0.5]

Seja

$$\frac{1}{z(z+1)} = \frac{A}{z} + \frac{B}{z+1}$$

$$\therefore \quad 1 = A(z+1) + Bz$$

$$1 = (A+B)z + A$$

$$\therefore \quad A+B = 0$$

$$A = 1$$

$$\therefore \quad // B = -A = -1 //$$

$$\therefore \quad \frac{1}{z(z+1)} = \frac{1}{z} - \frac{1}{z+1} \qquad \underline{[1.0]}$$

$$\therefore \int \frac{dz}{z(z+1)} = \int \frac{dz}{z} - \int \frac{dz}{(z+1)}$$

$$= \ln|z| - \ln|z+1| + C$$

$$= \ln\left|\operatorname{tg}\frac{x}{2}\right| - \ln\left|\operatorname{tg}\frac{x}{2}+1\right| + C$$

$$\underline{[1.5]}$$

3. $\int \sin^2 2t \cos^4 2t \, dt =$

$(u = 2t \ , \ du = 2dt)$

$= \int \sin^2 u \cos^4 u \ \frac{1}{2} du$

$= \frac{1}{2} \int (1 - \cos^2 u) \cos^4 u \ du$

$= \frac{1}{2} \int \cos^4 u \ du - \frac{1}{2} \int \cos^6 u \ du$ (*)

[0.5]

Mas:

$\int \cos^n u \ du = \int \underbrace{\cos^{n-1} u}_{v} \underbrace{\cos u \ du}_{dw}$

$(n \geq 1)$

$\left(\begin{array}{ll} v = \cos^{n-1} u & dw = \cos u \ du \\ dv = -(n-1) \cos^{n-2} u \sin u \ du & w = \sin u \end{array} \right)$

$= \sin u \cos^{n-1} u + (n-1) \int \cos^{n-2} u \ \sin^2 u \ du$

$= \sin u \ \cos^{n-1} u + (n-1) \int \cos^{n-2} u \ (1 - \cos^2 u) \ du$

$= \sin u \cos^{n-1} u + (n-1) \int \cos^{n-2} u \, du - (n-1) \int \cos^n u \ du$

Seja

$$n\int \cos^n u\, du = \sin u \cos^{n-1} u + (n-1)\int \cos^{n-2} u\, du$$

[1.0]

$$\int \cos^n u\, du = \frac{1}{n}\sin u \cos^{n-1} u + \frac{(n-1)}{n}\int \cos^{n-2} u\, du$$

Daí

$$\int \cos^4 u\, du = \frac{1}{4}\sin u \cos^3 u + \frac{3}{4}\int \cos^2 u\, du$$

[0.2]

$$= \frac{1}{4}\sin u \cos^3 u + \frac{3}{4}\left\{\frac{1}{2}\sin u \cos u + \frac{1}{2}\int du\right\}$$

$$= \frac{1}{4}\sin u \cos^3 u + \frac{3}{8}\sin u \cos u + \frac{3}{8} u$$

$$\int \cos^6 u = \frac{1}{6}\sin u \cos^5 u + \frac{5}{6}\int \cos^4 u \, du$$

$$= \frac{1}{6}\sin u \cos^5 u + \frac{5}{6}\left\{ \frac{1}{4}\sin u \cos^3 u + \frac{3}{8}\sin u \cos u + \frac{3}{8}u \right\}$$

[0-2]

$$= \frac{1}{6}\sin u \cos^5 u + \frac{5}{24}\sin u \cos^3 u + \frac{5}{16}\sin u \cos u + \frac{5}{16}u$$

[0-1]

$$(\star) = \frac{1}{2}\left\{ \frac{1}{4}\sin u \cos^3 u + \frac{3}{8}\sin u \cos u + \frac{3}{8}u \right\}$$

$$-\frac{1}{2}\left\{ \frac{1}{6}\sin u \cos^5 u + \frac{5}{24}\sin u \cos^3 u + \frac{5}{16}\sin u \cos u + \frac{5}{16}u \right\}$$

$$= \frac{1}{8}\sin u \cos^3 u + \frac{3}{16}\sin u \cos u + \frac{3}{16}u$$

$$-\frac{1}{12}\sin u \cos^5 u - \frac{5}{48}\sin u \cos^3 u - \frac{5}{32}\sin u \cos u - \frac{5}{32}u$$

$$= -\frac{1}{12}\sin u\cos^5 u + \frac{1}{48}\sin u\cos^3 u +$$

$$+ \frac{1}{32}\sin u\cos u + \frac{1}{32}u$$

$$\boxed{\begin{aligned}\int \sin^2 2t\cos^4 2t\,dt &= \\ &= -\frac{1}{12}\sin 2t\cos^5 2t + \frac{1}{48}\sin 2t\cos^3 2t \\ &\quad + \frac{1}{32}\sin 2t\cos 2t + \frac{1}{16}t + C\end{aligned}}$$

[0.1]

Outra Solução

$$\int \sin^2 2t \cos^4 2t \, dt =$$

$$= \int \frac{1-\cos 4t}{2} \cdot \left(\frac{1+\cos 4t}{2}\right)^2 dt$$

$$= \frac{1}{8}\int (1-\cos 4t)(1+2\cos 4t + \cos^2 4t)\, dt$$

$$= \frac{1}{8}\int \left(1 + 2\cos 4t + \cos^2 4t - \cos 4t - 2\cos^2 4t - \cos^3 4t\right) dt$$

$$= \frac{1}{8}\int \left(1 + \cos 4t - \cos^2 4t - \cos^3 4t\right) dt$$

$$= \frac{1}{8} t + \frac{1}{8}\frac{\sin 4t}{4} - \frac{1}{8}\int \cos^2 4t \, dt$$

$$- \frac{1}{8}\int \cos^3 4t \, dt$$

Mas :

$$-\frac{1}{8}\int \cos^2 4t\, dt = -\frac{1}{8}\int \frac{1+\cos 8t}{2}\, dt$$

$$= -\frac{1}{16}\int dt - \frac{1}{16}\int \cos 8t\, dt$$

$$= -\frac{1}{16} t - \frac{1}{16}\frac{\sin 8t}{8}$$

$$= -\frac{1}{16} t - \frac{1}{128}\sin 8t$$

$$-\frac{1}{8}\int \cos^3 4t\, dt = -\frac{1}{8}\int \cos^2 4t \cos 4t\, dt$$

$$= -\frac{1}{8}\int (1-\sin^2 4t)\cos 4t\, dt$$

$$= -\frac{1}{8}\int \cos 4t\, dt + \frac{1}{8}\int \sin^2 4t \cos 4t\, dt$$

$$= -\frac{1}{8}\frac{\sin 4t}{4} + \frac{1}{8}\frac{\sin^3 4t}{12}$$

$$= -\frac{1}{32}\sin 4t + \frac{1}{96}\sin^3 4t$$

$$\int \sin^2 2t \cos^4 2t \; dt =$$

$$= \underline{\frac{1}{8}t} + \cancel{\frac{1}{32}\sin 4t} \underline{- \frac{1}{16}t} - \frac{1}{128}\sin 8t$$

$$\cancel{- \frac{1}{32}\sin 4t} + \frac{1}{96}\sin^3 4t$$

$$= \frac{1}{96}\sin^3 4t - \frac{1}{128}\sin 8t + \frac{t}{16} + C \quad //$$

4. $\displaystyle\int \frac{2\,dx}{1+\sec\frac{x}{2}} = \left(\frac{x}{2}=u\right)$

$$= \int \frac{2\,du}{1+\sec u}$$

$$= \int \frac{2\cos u}{1+\cos u}\,du$$

$$= \int \frac{2\cos u\,(1-\cos u)}{(1+\cos u)(1-\cos u)}\,du$$

$$= \int \frac{2\cos u\,(1-\cos u)}{1-\cos^2 u}\,du$$

$$= \int \frac{2\cos u - 2\cos^2 u}{\sin^2 u}\,du$$

[1.0]

$$= 2\int \sin^{-2} u \cos u \;-2\int \cot g^2 u\,du$$

$$= 2\,(-1)\sin^{-1} u \;-2\int (\operatorname{cosec}^2 u - 1)\,du$$

$$= -2\operatorname{cosec} u \;-2\int \operatorname{cosec}^2 u\,du + 2\int du$$

[1.0]

$$= -2\operatorname{cosec} u + 2\cot g u + 2u$$

$$= -2\,\text{cosec}\,\frac{x}{2} + 2\,\text{cotg}\,\frac{x}{2} + x + C$$

outra solução

$$z = \text{tg}\,\frac{x}{4} \rightarrow dx = \frac{4\,dz}{1+z^2}, \quad \text{sen}\,\frac{x}{2} = \frac{2z}{1+z^2}, \quad \cos\frac{x}{2} = \frac{1-z^2}{1+z^2}$$

$$\therefore \int \frac{dx}{1+\sec\frac{x}{2}} = \int \frac{\frac{4\,dz}{1+z^2}}{1+\frac{1}{\cos\frac{x}{2}}} = \int \frac{\frac{4\,dz}{1+z^2}}{1+\frac{1+z^2}{1-z^2}} =$$

$$= \int \frac{\frac{4\,dz}{1+z^2}}{\frac{2}{1-z^2}} = \int \frac{4(1-z^2)\,dz}{2(1+z^2)}$$

$$\begin{array}{l|l} -z^2+1 & z^2+1 \\ \cline{2-2} +z^2+1 & -1 \\ \hline 0+2 & \end{array}$$

$$= 2\int -1\,dz + 2\int \frac{2\,dz}{z^2+1}$$

$$= -2z + 4\,\text{arc tg}\,z$$

$$= -2\,\text{tg}\,\frac{x}{4} + 4\,\frac{x}{4} + C$$

$$= -2\,\text{tg}\,\frac{x}{4} + x$$

outra solução: $\displaystyle\int \frac{dx}{1+\sec\frac{x}{2}} = \int \frac{dx}{1+\frac{1}{\cos\frac{x}{2}}} = \int \frac{\cos\frac{x}{2}}{1+\cos\frac{x}{2}}\,dx =$

$$= \int \frac{\cos\frac{x}{2}}{2\cos^2\frac{x}{4}}\,dx = \int \frac{2\cos^2\frac{x}{4}-1}{2\cos^2\frac{x}{4}}\,dx = \int dx - \frac{1}{2}\int \sec^2\frac{x}{4}\,dx$$

$$= x - \frac{1}{2}\int \sec^2 u\; 4\,du = x - 2\,\text{tg}\,u = x - 2\,\text{tg}\,\frac{x}{4}$$

5. $$\int \frac{2x^2 - x + 2}{x^5 + 2x^3 + x}\, dx$$

$$x^5 + 2x^3 + x = x(x^4 + 2x^2 + 1)$$
$$= x\underbrace{(x^2+1)^2}_{\text{irredutível}}$$

$$\frac{2x^2 - x + 2}{x^5 + 2x^3 + x} = \frac{2x^2 - x + 2}{x(x^2+1)^2} =$$

$$= \frac{A}{x} + \frac{Bx + C}{x^2+1} + \frac{Dx + E}{(x^2+1)^2}$$

[0.5]

$$2x^2 - x + 2 = A(x^2+1)^2 + (Bx + C)\overbrace{x(x^2+1)}^{x^3+x} + (Dx+E)x$$
$$= A(x^4 + 2x^2 + 1) + Bx^4 + Bx^2 + Cx^3 + Cx +$$
$$+ Dx^2 + Ex$$

$$2x^2 - x + 2 = (A+B)x^4 + Cx^3 + (2A + B + D)x^2 + (C+E)x + A$$

$$\begin{cases} A+B=0 \implies \underline{B=-A=-2} \\ \underline{C=0} \\ 2A+B+D=2 \\ C+E=-1 \implies \underline{E=-1} \\ \underline{A=2} \end{cases}$$

$$\therefore \quad 2A+B+D=2$$

$$4+(-2)+D=2$$

$$\therefore \quad \underline{D=0}$$

$$\therefore \quad \frac{2x^2-x+2}{x^5+2x^3+x} = \frac{2}{x} + \frac{-2x}{x^2+1} + \frac{-1}{(x^2+1)^2}$$

[1.0]

$$\int \frac{2x^2-x+2}{x^5+2x^3+x}\,dx = \int \frac{2}{x}\,dx - 2\int \frac{x}{x^2+1}\,dx - \int \frac{dx}{(x^2+1)^2}$$

$$= 2\ln|x| - 2\,\frac{1}{2}\ln(x^2+1) - \int \frac{dx}{(x^2+1)^2}$$

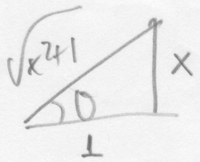
[1.5]

Mas

$$\int \frac{dx}{(x^2+1)^2} = \int \frac{\sec^2\theta \, d\theta}{\sec^4\theta}$$

$$\left(\begin{array}{l} x = tg\theta \\ dx = \sec^2\theta \, d\theta \end{array}\right) = \int \frac{d\theta}{\sec^2\theta}$$

$$= \int \cos^2\theta \, d\theta$$



$$= \int \frac{1+\cos 2\theta}{2} \, d\theta$$

$$= \frac{1}{2}\theta + \frac{\sin 2\theta}{4}$$

$$= \frac{1}{2}\theta + \frac{2}{4}\sin\theta\cos\theta$$

$$= \frac{1}{2}\,\mathrm{arctg}\, x + \frac{1}{2}\frac{x}{\sqrt{x^2+1}}\,\frac{1}{\sqrt{x^2+1}}$$

$$= \frac{1}{2}\,\mathrm{arctg}\, x + \frac{x}{2(x^2+1)}$$

$\therefore$

$$\boxed{\int \frac{2x^2-x+2}{x^5+2x^3+x}\,dx = 2\ln|x| - \ln(x^2+1) - \frac{1}{2}\,\mathrm{arctg}\,x - \frac{x}{2(x^2+1)} + C}$$

[2.5]