Geometria Analítica - Prova 1

2,0 1. Determine todas as matrizes X tal que se tem

$$X^2 = \begin{pmatrix} 3 & 4 \\ 2 & 3 \end{pmatrix}$$

1,0 2. Calcule o valor do determinante da matriz

$$\begin{pmatrix} a & b & c & d \\ b & c & d & a \\ c & d & a & b \\ d & a & b & c \end{pmatrix}$$

sabendo que $a + b + c + d = 0$

2,0 3. Resolva o sistema usando o método de Gauss-Jordan

$$\begin{cases} x + 2y - z = 3 \\ 2x + y + z = 3 \\ x + y + 2z = 4 \end{cases}$$

1,0 4. Calcule a área do quadrilátero ABCD sabendo que $A(4, 0, 0)$, $B(0, 0, 2)$, $C(0, 3, 0)$, $D(4, 3, -2)$

1.

$$X = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$X^2 = \underbrace{\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}} = \begin{pmatrix} 3 & 4 \\ 2 & 3 \end{pmatrix}$$

$$\begin{pmatrix} a^2+bc & ab+bd \\ ac+cd & bc+d^2 \end{pmatrix} = \begin{pmatrix} 3 & 4 \\ 2 & 3 \end{pmatrix}$$

$$\therefore \begin{cases} a^2+bc = 3 & (1) \\ ab+bd = 4 & (2) \\ ac+cd = 2 & (3) \\ bc+d^2 = 3 & (4) \end{cases}$$

0,5

De (1) e (4): $a^2 + \cancel{bc} = \cancel{bc} + d^2$

$$a^2 = d^2 \quad \therefore \quad \underline{a = \pm d} \quad (*)$$

(2): $ab + bd = 4$

$$b(a+d) = 4 \Longrightarrow a+d \neq 0 \quad \therefore \boxed{a = d} \quad (*)$$

$$\therefore b \cdot 2a = 4$$

$$\boxed{ab = 2}$$

(3) : $ac + cd = 2$

$$c(a+d) = 2$$

$$\cancel{2}ac = \cancel{2}$$

$$\boxed{ac = 1}$$

Temos então :

$$\begin{cases} a = d \\ ab = 2 \xrightarrow{a \neq 0} b = \frac{2}{a} \\ ac = 1 \longrightarrow c = \frac{1}{a} \end{cases}$$

Daí :

$$a^2 + bc = 3$$

$$a^2 + \frac{2}{a}\frac{1}{a} = 3$$

$$a^2 + \frac{2}{a^2} = 3$$

$$a^4 + 2 = 3a^2$$

$$a^4 - 3a^2 + 2 = 0$$

$$a^2 = 3 \pm \sqrt{9-8} \;/ 2$$

$$a^2 = \frac{3 \pm 1}{2} \begin{cases} \nearrow 2 \longrightarrow a = \pm\sqrt{2} \\ \searrow 1 \longrightarrow a = \pm 1 \end{cases}$$

1,0

Podemos ter então :

$$a = 1 \quad : \quad X = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}$$

$$a = -1 \quad : \quad X = \begin{pmatrix} -1 & -2 \\ -1 & -1 \end{pmatrix}$$

$$a = \sqrt{2} \quad : \quad X = \begin{pmatrix} \sqrt{2} & \sqrt{2} \\ \frac{\sqrt{2}}{2} & \sqrt{2} \end{pmatrix}$$

$$a = -\sqrt{2} \quad : \quad X = \begin{pmatrix} -\sqrt{2} & -\sqrt{2} \\ -\frac{\sqrt{2}}{2} & -\sqrt{2} \end{pmatrix}$$

0.7

2.

$$A = \begin{pmatrix} a & b & c & d \\ b & c & d & a \\ c & d & a & b \\ d & a & b & c \end{pmatrix}$$

Seja

$$A \xrightarrow{L_1 \to L_1 + L_2} A_1 = \begin{pmatrix} a+b & b+c & c+d & d+a \\ b & c & d & a \\ c & d & a & b \\ d & a & b & c \end{pmatrix}$$

$$\det A_1 = \det A \quad (*)$$

$$A_1 \xrightarrow{L_1 \to L_1 + L_3} A_2 = \begin{pmatrix} a+b+c & b+c+d & c+d+a & d+a+b \\ b & c & d & a \\ c & d & a & b \\ d & a & b & c \end{pmatrix}$$

$$\det A_2 = \det A_1 \overset{(*)}{=} \det A$$

$\therefore \quad \det A_2 = \det A \quad (**)$

$$A_2 \xrightarrow{L_1 \to L_1 + L_4} A_3 = \begin{pmatrix} a+b+c+d & b+c+d+a & c+d+a+b & d+a+b+c \\ b & c & d & a \\ c & d & a & b \\ d & a & b & c \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 & 0 \\ b & c & d & a \\ c & d & a & b \\ d & a & b & c \end{pmatrix}$$

$\therefore \quad \det A_3 = \det A_2 \overset{(**)}{=} \det A$

$\therefore \quad \det A = \det A_3 = 0$ pois A_3 tem $\longrightarrow$

uma linha toda nula.

$\therefore \quad \underline{\det A = 0}$

3. $\begin{cases} x + 2y - z = 3 \\ 2x + y + z = 3 \\ x + y + 2z = 4 \end{cases}$

$$A = \begin{pmatrix} 1 & 2 & -1 & 3 \\ 2 & 1 & 1 & 3 \\ 1 & 1 & 2 & 4 \end{pmatrix}$$

$$\downarrow L_2 \to L_2 - 2L_1$$

$$A_1 = \begin{pmatrix} 1 & 2 & -1 & 3 \\ 0 & -3 & 3 & -3 \\ 1 & 1 & 2 & 4 \end{pmatrix}$$

$$\downarrow L_3 \to L_3 - L_1$$

$$A_2 = \begin{pmatrix} 1 & 2 & -1 & 3 \\ 0 & -3 & 3 & -3 \\ 0 & -1 & 3 & 1 \end{pmatrix}$$

$$\downarrow L_3 \to L_3 - \frac{1}{3} L_2$$

$$A_3 = \begin{pmatrix} 1 & 2 & -1 & 3 \\ 0 & -3 & 3 & -3 \\ 0 & 0 & 2 & 2 \end{pmatrix}$$

1.0

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$$x + 2y - z = 3 \quad (*)$$

$$-3y + 3z = -3 \quad (**)$$

$$2z = 2 \Rightarrow \underline{\underline{z = 1}}$$

$$(**) : \quad -3y + 3 = -3$$

$$-3y = -6$$

$$\underline{\underline{y = 2}}$$

$$(*) : \quad x + 4 - 1 = 3$$

$$x + 3 = 3$$

$$\underline{\underline{x = 0}}$$

$$\therefore \quad S = \{ (x, y, z) = (0, 2, 1) \}$$

1.0

Outra Solução

$$\begin{cases} x + 2y - z = 3 \\ 2x + y + z = 3 \\ x + y + 2z = 4 \end{cases}$$

$$A = \begin{pmatrix} 1 & 2 & -1 & 3 \\ 2 & 1 & 1 & 3 \\ 1 & 1 & 2 & 4 \end{pmatrix}$$

$$\downarrow L_2 \to L_2 - 2L_1$$

$$A_1 = \begin{pmatrix} 1 & 2 & -1 & 3 \\ 0 & -3 & 3 & -3 \\ 1 & 1 & 2 & 4 \end{pmatrix}$$

$$\downarrow L_3 \to L_3 - L_1$$

$$A_2 = \begin{pmatrix} 1 & 2 & -1 & 3 \\ 0 & -3 & 3 & -3 \\ 0 & -1 & 3 & 1 \end{pmatrix}$$

$$\downarrow L_3 \to L_3 - \tfrac{1}{3} L_2$$

$$A_3 = \begin{pmatrix} 1 & 2 & -1 & 3 \\ 0 & -3 & 3 & -3 \\ 0 & 0 & 2 & 2 \end{pmatrix}$$

$$\downarrow L_1 \to L_1 + \tfrac{2}{3} L_2$$

$$A_4 = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & -3 & 3 & -3 \\ 0 & 0 & 2 & 2 \end{pmatrix}$$

A_4

$$\downarrow L_1 = L_1 - \tfrac{1}{2} L_3$$

$$A_5 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -3 & 3 & -3 \\ 0 & 0 & 2 & 2 \end{pmatrix}$$

$$\downarrow L_2 \to -\tfrac{1}{3} L_2, \quad L_3 \to \tfrac{1}{2} L_3$$

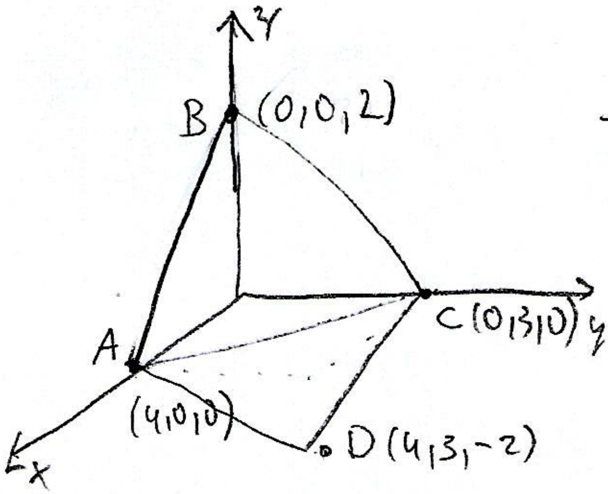
$$A_6 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

$$\downarrow L_2 \to L_2 + L_3$$

$$A_7 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

$$\therefore \begin{cases} x = 0 \\ y = 2 \\ z = 1 \end{cases}$$

4.



$S_{ABCD} = S_{ABC} + S_{ACD}$

$\vec{AB} = (-4,0,2)$
$\vec{DC} = (-4,0,2)$

$\vec{AD} = (0,3,-2)$
$\vec{BC} = (0,3,-2)$

$$\begin{cases} S_{ABC} = \frac{1}{2} \left| \vec{AB} \times \vec{AC} \right| \\ S_{ACD} = \frac{1}{2} \left| \vec{AC} \times \vec{AD} \right| \end{cases}$$

0.5

A figura é um paralelogramo

$S = \left| \vec{AB} \times \vec{AD} \right|$

Mas

$$\begin{cases} \vec{AB} = (-4,0,2) \\ \vec{AC} = (-4,3,0) \\ \vec{AD} = (0,3,-2) \end{cases}$$

Daí,

$$\vec{AB} \times \vec{AC} = \det \begin{pmatrix} \hat{i} & \hat{j} & \hat{k} \\ -4 & 0 & 2 \\ -4 & 3 & 0 \end{pmatrix}$$

$$= -8\hat{j} - 12\hat{k} - 6\hat{i}$$

$$= (-6,-8,-12)$$

$$\begin{array}{r|l} 244 & 2 \\ 122 & 2 \\ 61 & \end{array}$$

$$\left| \vec{AB} \times \vec{AC} \right| = \sqrt{36+64+144} = \sqrt{244} = 2\sqrt{61}$$

$$\therefore \; S_{ABC} = \frac{1}{2} \, 2\sqrt{61} = \sqrt{61}$$

0,5

$$\overrightarrow{AC} \times \overrightarrow{AD} = \det \begin{pmatrix} \hat{i} & \hat{j} & \hat{k} \\ -4 & 3 & 0 \\ 0 & 3 & -2 \end{pmatrix}$$

$$= -6\hat{i} - 12\hat{k} - 8\hat{j}$$

$$= (-6, -8, -12)$$

$$|\overrightarrow{AC} \times \overrightarrow{AD}| = \sqrt{36 + 64 + 144} = 2\sqrt{61}$$

$\therefore$ // $S_{ACD} = \frac{1}{2} \, 2\sqrt{61} = \sqrt{61}$ //   0.5

Donc

$$S_{ABCD} = \sqrt{61} + \sqrt{61}$$

$\therefore$ $\boxed{S_{ABCD} = 2\sqrt{61}}$ 0.5

$$(\overrightarrow{AB} \times \overrightarrow{AD}) = \det \begin{pmatrix} \hat{i} & \hat{j} & \hat{k} \\ -4 & 0 & 2 \\ 0 & 3 & -2 \end{pmatrix}$$

244	2
122	2
61	61

$$= -12\hat{k} + 6\hat{i} - 8\hat{j}$$

$$= (6, -8, -12)$$

$$S = |\overrightarrow{AB} \times \overrightarrow{AD}| = \sqrt{36 + 64 + 144}$$

$$= \sqrt{100 + 144} = \sqrt{244}$$

$$= 2\sqrt{61}$$ //