Ex.

$$\begin{cases} f(x) = 2x^2+5 \ ; & \text{Dom } f = \mathbb{R} \\ g(x) = 4-7x \ ; & \text{Dom } g = \mathbb{R} \end{cases}$$

$\rightarrow$ $(f \circ g)(x) = f(g(x)) = 2\,g(x)^2+5 = 2\,(4-7x)^2+5$

$= 2\,(16-56x+49x^2)+5$

$= 32-112x+98x^2+5$

$= 98x^2-112x+37$ $\rightarrow$ definida para toda $x \in \mathbb{R}$

Mas

$\text{Dom } f \circ g \subset \text{Dom } g = \mathbb{R}$

$\therefore$ $\text{Dom } f \circ g = \mathbb{R}$

$\therefore$

$$\boxed{\begin{array}{l} (f \circ g)(x) = 98x^2-112x+37 \\ \text{Dom } f \circ g = \mathbb{R} \end{array}}$$

$\rightarrow$ $(g \circ f)(x) = g(f(x)) = 4-7f(x) = 4-7\,(2x^2+5)$

$= 4-14x^2-35$

$= -14x^2-31$ $\rightarrow$ definido para todo $x \in \mathbb{R}$

Mas $\text{Dom } g \circ f \subset \text{Dom } f = \mathbb{R}$

$\therefore$

$$\boxed{\begin{array}{l} (g \circ f)(x) = -14x^2-31 \\ \text{Dom } g \circ f = \mathbb{R} \end{array}}$$

Ex.

$$\begin{cases} f(x) = \sqrt[3]{x^2+1} & ; \quad \text{Dom } f = \mathbb{R} \\ g(x) = x^3+1 & ; \quad \text{Dom } g = \mathbb{R} \end{cases}$$

$\rightarrow$ $(f \circ g)(x) = f(g(x)) = \sqrt[3]{g(x)^2+1} =$

$$= \sqrt[3]{(x^3+1)^2+1} = \sqrt[3]{x^6+2x^3+1+1}$$

$$= \sqrt[3]{x^2+2x^3+2} \rightarrow \text{definida para todo } x \in \mathbb{R}$$

Mas

$$\text{Dom } f\circ g \subset \text{Dom } g = \mathbb{R}$$

$$\therefore \text{Dom } f\circ g = \mathbb{R}$$

$$\therefore \boxed{\begin{array}{l} (f\circ g)(x) = \sqrt[3]{x^2+2x^3+2} \\ \text{Dom } f\circ g = \mathbb{R} \end{array}}$$

$\rightarrow$ $(g\circ f)(x) = g(f(x)) = f(x)^3+1 = \left(\sqrt[3]{x^2+1}\right)^3+1$

$$= x^2+1+1 = x^2+2 \rightarrow \text{definida para todo } x \in \mathbb{R}$$

Mas $\text{Dom } g\circ f \subset \text{Dom } f = \mathbb{R} \quad \therefore \text{Dom } g\circ f = \mathbb{R}$

$$\therefore \boxed{(g\circ f)(x) = x^2+1 \ , \ \text{Dom } g\circ f = \mathbb{R}}$$

Ex.

$$\begin{cases} f(x) = x^2 - 4x + 1 \;, & \text{Dom } f = \mathbb{R} \\ g(x) = x^2 - 1 \;, & \text{Dom } g = \mathbb{R} \end{cases}$$

$\rightarrow$ $(f\circ g)(x) = f(g(x)) = g(x)^2 - 4\,g(x) + 1$

$= (x^2-1)^2 - 4(x^2-1) + 1$

$= x^4 - 2x^2 + 1 - 4x^2 + 4 + 1$

$= x^4 - 6x^2 + 6 \rightarrow$ definida para todo $x \in \mathbb{R}$

Mas

$\text{Dom } f\circ g \subset \text{Dom } g = \mathbb{R}$

$\therefore \text{Dom } f\circ g = \mathbb{R}$

$$\therefore \boxed{\begin{array}{l} (f\circ g)(x) = x^4 - 6x^2 + 6 \\ \text{Dom } f\circ g = \mathbb{R} \end{array}}$$

$\rightarrow$ $(g\circ f)(x) = g(f(x)) = f(x)^2 - 1 =$

$= (x^2-4x+1)^2 - 1 = x^4 + 16x^2 + \cancel{1} - 8x^3 + 2x^2 - 8x - \cancel{1}$

$= x^4 - 8x^3 + 18x^2 - 8x \rightarrow$ definido para $x \in \mathbb{R}$

Mas $\text{Dom } g\circ f \subset \text{Dom } f = \mathbb{R}$

$\therefore$ Dom $g \circ f = \mathbb{R}$

$\therefore$ $\boxed{\begin{array}{l} (g \circ f)(x) = x^4 - 8x^3 + 18x^2 - 8x \\ \text{Dom } g \circ f = \mathbb{R} \end{array}}$

Ex.:

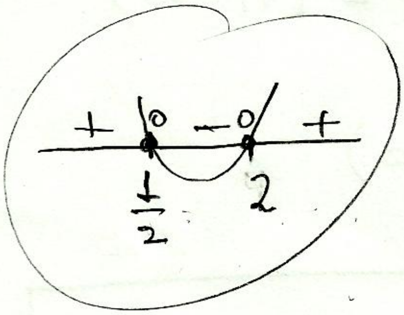
$$\begin{cases} f(x) = \sqrt{x-1} \ ; & \text{Dom } f = [1, +\infty) \\ g(x) = 2x^2 - 5x + 3 \ ; & \text{Dom } g = \mathbb{R} \end{cases}$$

$$\rightarrow (f \circ g)(x) = f(g(x)) = \sqrt{g(x) - 1}$$

$$= \sqrt{2x^2 - 5x + 3 - 1}$$

$$= \sqrt{2x^2 - 5x + 2} \longrightarrow$$ está definido para x satisfazendo $2x^2 - 5x + 2 \geq 0$

$\therefore$



$$2x^2 - 5x + 2 = 0$$

$$\therefore x = \frac{5 \pm \sqrt{25 - 16}}{4} = \frac{5 \pm \sqrt{9}}{4}$$

$$= \frac{5 \pm 3}{4} \begin{matrix} \nearrow 2 \\ \searrow \frac{1}{2} \end{matrix}$$

i.e. devemos ter $x \in (-\infty, \frac{1}{2}] \cup [2, +\infty)$

Mas

$$\text{Dom } f \circ g \subset \text{Dom } g = \mathbb{R}$$

$$\therefore \quad \text{Dom } f \circ g = (-\infty, \tfrac{1}{2}] \cup [2, +\infty)$$

$\therefore$

$$\boxed{\begin{array}{l} (f \circ g)(x) = \sqrt{2x^2 - 5x + 2} \\ \text{Dom } f \circ g = (-\infty, \frac{1}{2}] \cup [2, +\infty) \end{array}}$$

$$\rightarrow (g\circ f)(x) = g(f(x)) = 2f(x)^2 - 5f(x) + 3$$

$$= 2(\sqrt{x-1})^2 - 5\sqrt{x-1} + 3$$

$$= 2(x-1) - 5\sqrt{x-1} + 3$$

$$= 2x - \underbrace{2 - 5\sqrt{x-1} + 3}$$

$$= 2x - 5\sqrt{x-1} + 1 \quad (*)$$

$(*)$ está definida para $x \geq 1$

Mas $\quad D_m\, g\circ f \subset D_m\, f = [1, +\infty)$

$\therefore \quad D_m\, g\circ f = [1, +\infty)$

$\therefore$

$$\boxed{\begin{array}{l}(g\circ f)(x) = 2x - 5\sqrt{x-1} + 1 \\ D_m\, g\circ f = [1, +\infty)\end{array}}$$

▲

Ex.:

$$f(x) = \begin{cases} 4x-3 & \text{se } x \geq 0 \\ x^2-3x+2 & \text{se } x < 0 \end{cases}$$

$$g(x) = \begin{cases} x+1 & \text{se } x > 2 \\ 1-x^2 & \text{se } x \leq 2 \end{cases}$$

$$\rightarrow (f \circ g)(x) = f(g(x)) = \begin{cases} 4g(x)-3 & \text{se } g(x) \geq 0 \\ g(x)^2-3g(x)+2 & \text{se } g(x) < 0 \end{cases}$$

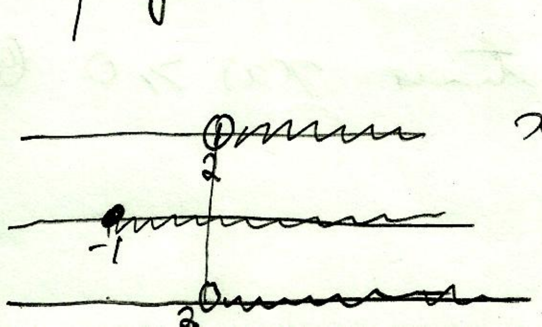
Devemos analisar os valores de x que resultam $g(x) \geq 0$ e $g(x) < 0$.

Seja $\underline{\underline{x > 2}}$.

Temos nesta faixa $g(x) = x+1$.

Daí $\begin{cases} g(x) \geq 0 \quad \therefore \quad x+1 \geq 0 \quad \therefore \quad x \geq -1 \\ g(x) < 0 \quad \therefore \quad x+1 < 0 \quad \therefore \quad x < -1 \end{cases}$

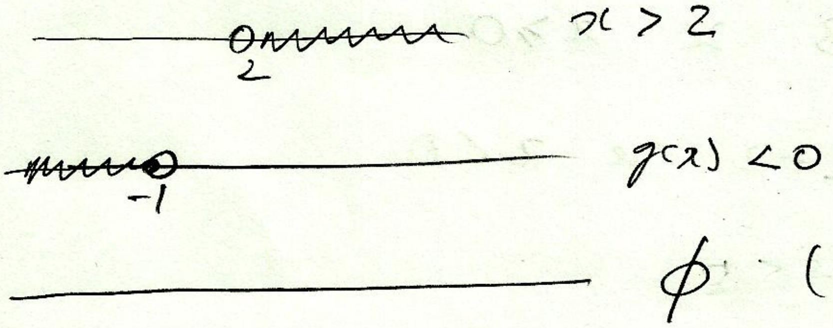
Daí:



$x > 2$

$g(x) \geq 0$

$x > 2$ e $g(x) \geq 0$ (*)



ϕ $(x > 2 \text{ e } g(x) < 0)$ (**)

isto é:

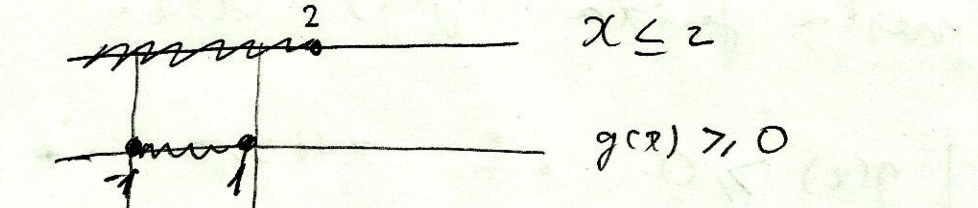
se $x > 2$ temos $g(x) \geq 0$ (3*).

Seja $x \leq 2$.

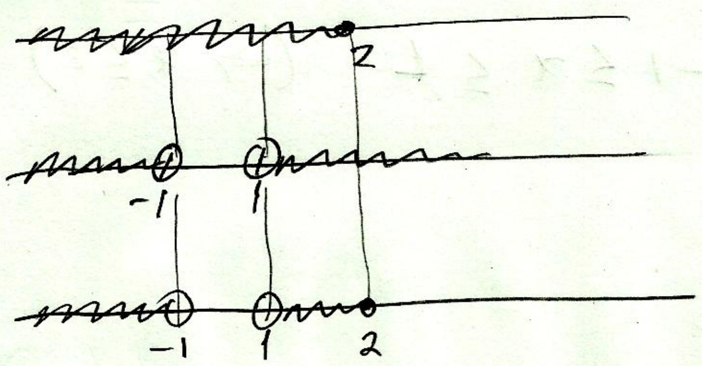
Nesta faixa temos $g(x) = 1 - x^2$

Daí
$$\begin{cases} g(x) \geq 0 & \therefore\ 1 - x^2 \geq 0 \ \therefore\ -1 \leq x \leq 1 \\ g(x) < 0 & \therefore\ 1 - x^2 < 0 \\ & \therefore\ x \in (-\infty, -1) \cup (1, +\infty) \end{cases}$$

Daí



i.e. se $-1 \leq x \leq 1$ temos $g(x) \geq 0$ (4*)



$x \leq 2$

$g(x) < 0$

$x < -1$ ou $1 < x \leq 2$

i.e.,

se $x \in (-\infty, -1) \cup (1, 2]$ temos $g(x) < 0$ (5*)

De (3*), (4*) e (5*) temos:

$$\left.\begin{array}{lll} g(x) \geq 0 & \text{se} & x \in [-1, 1] \cup (2, +\infty) \\ g(x) < 0 & \text{se} & x \in (-\infty, -1) \cup (1, 2] \end{array}\right\}$$

Com isso, reescrevemos fog na forma

$$(f \circ g)(x) = \begin{cases} 4g(x) - 3 & \text{se} \quad -1 \leq x \leq 1 \quad (\Rightarrow x \leq 2) \\ 4g(x) - 3 & \text{se} \quad 2 < x \quad (\Rightarrow x > 2) \\ g(x)^2 - 3g(x) + 2 & \text{se} \quad 1 < x \leq 2 \quad (\Rightarrow x \leq 2) \\ g(x)^2 - 3g(x) + 2 & x < -1 \quad (\Rightarrow x \leq 2) \end{cases}$$

Mos

$(f\circ g)(x) = 4\underbrace{g(x)}-3$ si $-1 \le x \le 1$ $(\Rightarrow x \le 2)$

$= 4(1-x^2)-3$

$= 4 - 4x^2 - 3$

$= -4x^2+1$ si $-1 \le x \le 1$

$(f\circ g)(x) = 4g(x)-3$ si $2 < x$ $(\Rightarrow x > 2)$

$= 4(x+1)-3$

$= 4x+4-3$

$= 4x+1$ si $x > 2$

$(f\circ g)(x) = g(x)^2 - 3g(x) + 2$ si $1 < x \le 2$ $(\Rightarrow x \le 2)$

$= (1-x^2)^2 - 3(1-x^2) + 2$

$= \cancel{1} - 2x^2 + x^4 - \cancel{3} + 3x^2 + \cancel{2}$

$= x^4 + x^2$ si $1 < x \le 2$

$f\circ g(x) = g(x)^2 - 3g(x) + 2$ si $x < -1$ $(\Rightarrow x \le 2)$

$= x^4 + x^2$ si $x < -1$

$$(f\circ g)(x) = \begin{cases} x^4+x^2 & \text{se} \quad x < -1 \\ -4x^2+1 & \text{se} \quad -1 \leq x \leq 1 \\ x^4+x^2 & \text{se} \quad 1 < x \leq 2 \\ 4x+1 & \text{se} \quad x > 2 \end{cases}$$