Cálculo A

Funções trigonométricas e trigonométricas inversas

1. Encontre o valor numérico das expressões a seguir

$a)$ $\arcsin \frac{1}{2}$	$b)$ $\arccos 1$	$c)$ $\operatorname{arccsc}(-1)$	$d)$ $\arctan 0$	$e)$ $\operatorname{arccot}(-\sqrt{3})$
$f)$ $\arctan(-\sqrt{3})$	$g)$ $\operatorname{arccsc}\sqrt{2}$	$h)$ $\operatorname{arcsec} 2$	$i)$ $\operatorname{arccsc} 2\sqrt{3}/3$	$j)$ $\operatorname{arcsec}(-2)$
$k)$ $\operatorname{arcsec}(-2\sqrt{3}/3)$	$l)$ $\arcsin 0$	$m)$ $\arcsin -\frac{1}{2}$	$n)$ $\arccos(-\sqrt{3}/2)$	$o)$ $\arctan 1$

2. Encontre o valor numérico das expressões a seguir

$a)$ $\sin(\arccos \frac{1}{2})$	$b)$ $\tan(\arcsin \sqrt{3}/2)$	$c)$ $\sec(\arccos \sqrt{3}/2)$	$d)$ $\csc(\arctan(-1))$
$e)$ $\sin(\arcsin(-\frac{1}{2}))$	$f)$ $\csc(\operatorname{arccot}(-\sqrt{3}))$	$g)$ $\csc(\operatorname{arcsec}\sqrt{2})$	$h)$ $\arcsin(\cos \pi/6)$
$i)$ $\operatorname{arccot}(\tan \pi/3)$	$j)$ $\arctan(\tan 0)$		

3. Determinar o domínio das funções

   (a) $f(x) = \frac{\cot 2x}{\sin \frac{x}{3}}$

   (b) $f(x) = \sqrt{\cos x}$

   (c) $f(x) = \left(\sin x - 2\sin^2 x\right)^{-3/4}$

   (d) $f(x) = \arccos(3 - x)$

   (e) $f(x) = \arcsin\left(\frac{1}{2}x - 1\right) + \arccos\left(1 - \frac{1}{2}x\right)$

   (f) $f(x) = 3\arcsin\sqrt{\frac{3x-1}{2}}$

   (g) $f(x) = \arccos \frac{1}{x-1}$

4. Determinar a imagem das funções

   (a) $f(x) = 1 - 2|\cos x|$

   (b) $f(x) = \sin x + \sin\left(x + \frac{\pi}{3}\right)$

   (c) $f(x) = \sin^4 x + \cos^4 x$

   (d) $f(x) = \frac{1+\sin x}{\sin x}$

   (e) $f(x) = \arccos |x|$

   (f) $f(x) = \pi - |\arctan x|$

   (g) $f(x) = \cos \arcsin x$

   (h) $f(x) = \arctan \frac{2x}{x^2+1}$

   (i) $f(x) = \arccos \frac{3x-1}{2}$ com $x \in [0, 1]$

5. Seja $f : A \to [0, 1]$ com $f(x) = \sin^2 2x$. Determine o maior subconjunto $A \subset \mathbb{R}$ de modo que f admita inversa.

6. Mostre que

   a) $\sec(\arctan x) = \sqrt{1+x^2}$ b) $\sin(\text{arccsc}\, x) = \frac{1}{x}$ c) $\cos(2\arcsin x) = 1-2x^2$
   d) $\sin(2\arcsin x) = 2x\sqrt{1-x^2}$
   Obs.: É possível termos $\sec(\arctan x) = -\sqrt{1+x^2}$ e $\sin(2\arcsin x) = -2x\sqrt{1-x^2}$? Explique.

7. a) Mostre que
$$\arcsin x + \arcsin y = \arcsin\left(x\sqrt{1-y^2} + y\sqrt{1-x^2}\,\right)$$
   desde que o valor da expressão do lado esquerdo esteja entre $[-\frac{\pi}{2}, \frac{\pi}{2}]$. (Tal condição é usada apenas para garantir que pode-se escrever o lado esquerdo como o arco seno da expressão dada do lado direito)

   b) Mostre que
$$\arctan \frac{x}{\sqrt{1-x^2}} = \arcsin x, \quad \text{com} \quad -1 < x < 1$$

   c) Mostre que
$$\arctan x + \arctan y = \arctan \frac{x+y}{1-xy} \quad \text{para} \quad xy \neq 1$$
   considerando que o valor da expressão do lado esquerdo esteja entre $[-\frac{\pi}{2}, \frac{\pi}{2}]$.

   d) Sejam a, b, c números satisfazendo $bc = 1 + a^2$. Mostre que
$$\arctan \frac{1}{a+b} + \arctan \frac{1}{a+c} = \arctan \frac{1}{a}$$
   desde que a expressão do lado esquerdo esteja entre $(-\frac{\pi}{2}, \frac{\pi}{2})$, e que tenhamos $a + b \neq 0, \quad a + c \neq 0, \quad a \neq 0$.

   e) Mostre que
$$\begin{aligned}\arcsin\left(\frac{x}{3} - 1\right) &= \frac{\pi}{2} - 2\arcsin\sqrt{1 - \frac{x}{6}} \\ \arcsin\left(\frac{x}{3} - 1\right) &= 2\left(\arcsin\frac{\sqrt{x}}{\sqrt{6}}\right) - \frac{\pi}{2}\end{aligned}$$

   f) Mostre que existe uma constante c tal que se tem
$$\arcsin x + \arccos x = c, \quad \text{com} \quad -1 \leq x \leq 1$$

   g) Seja $f(x) = \arctan x + \arctan \frac{1}{x}$. Mostre que $f(x)$ é constante em cada um dos intervalos $(-\infty, 0)$ e $(0, \infty)$. Encontre as constantes.

8. Mostre que $\arcsin \frac{m-1}{m+1} = \arccos \frac{2\sqrt{m}}{m+1}$ $(m \geq 0)$

9. Determine $x \in (0, 1)$ tal que $\arcsin x + \arcsin 2x = \frac{\pi}{2}$

10. Se $a \in \mathbb{R}, a > 0$ e $0 \leq \arcsin \frac{a-1}{a+1} \leq \frac{\pi}{2}$ mostre que $\tan\left(\arcsin \frac{a-1}{a+1} + \arctan \frac{1}{2\sqrt{a}}\right) = \frac{2a\sqrt{a}}{3a+1}$

11. Determine a solução de $\arctan x + \arctan \frac{x}{x+1} = \frac{\pi}{4}$ $(x \neq -1)$

12. Determine a solução de

$$\sec\Big(\arctan \frac{1}{1+e^x} - \arctan(1-e^x)\Big) = \frac{\sqrt{5}}{2}$$

13. Determine os valores de $a \in (-\frac{\pi}{2}, \frac{\pi}{2})$ para os quais existe $x \in \mathbb{R}$ solução de

$$\arctan\Big(\sqrt{2} - 1 + \frac{e^x}{2}\Big) + \arctan\Big(\sqrt{2} - 1 - \frac{e^x}{2}\Big) = a$$

14. Seja $x = \arcsin \frac{b}{a}$ com $|a| > |b|$, $0 \leq x \leq \frac{\pi}{2}$. Mostre que $x = \frac{1}{2} \arcsin \frac{2b\sqrt{a^2-b^2}}{a|a|}$

15. Determine um intervalo I que contêm todas as soluções de

$$\arctan \frac{1+x}{2} + \arctan \frac{1-x}{2} \geq \frac{\pi}{4}$$

16. Mostre que

$$\sin\Big(2\,\text{arccot}\frac{4}{3}\Big) + \cos\Big(2\text{arccsc}\frac{5}{4}\Big) = \frac{17}{25}$$

Respostas

1.

$a)\ \pi/6$ $b)\ 0$ $c)\ -\frac{\pi}{2}$ $d)\ 0$ $e)\ \frac{5\pi}{6}$ $f)\ -\frac{\pi}{3}$ $g)\ \frac{\pi}{4}$ $h)\ \frac{\pi}{3}$ $i)\ \frac{\pi}{3}$ $j)\ \frac{4\pi}{3}$ $k)\ \frac{7\pi}{6}$ $l)\ 0$
$m)\ -\frac{\pi}{6}$ $n)\ \frac{5\pi}{6}$ $o)\ \frac{\pi}{4}$

2.

$a)\ \frac{\sqrt{3}}{2}$ $b)\ \sqrt{3}$ $c)\ \frac{2}{\sqrt{3}}$ $d)\ -\sqrt{2}$ $e)\ -\frac{1}{2}$ $f)\ 2$ $g)\ \sqrt{2}$ $h)\ \frac{\pi}{3}$ $i)\ \frac{\pi}{6}$ $j)\ 0$

3. (a) $\{x \in \mathbb{R} : x \neq n\pi/2; n \in \mathbb{Z}\}$
   (b) $\cup_{n\in\mathbb{Z}}[-\frac{\pi}{2} + 2\pi n, \frac{\pi}{2} + 2\pi n]$
   (c) $\cup_{n\in\mathbb{Z}}(2\pi n, \frac{\pi}{6} + 2\pi n) \cup (\frac{5\pi}{6} + 2\pi n, \pi + 2\pi n)$
   (d) $[2, 4]$
   (e) $[0, 4]$
   (f) $[\frac{1}{3}, 1]$
   (g) $(-\infty, 0] \cup [2, \infty)$

4. (a) $[-1, 1]$
   (b) $[-\sqrt{3}, \sqrt{3}]$
   (c) $[\frac{1}{2}, 1]$
   (d) $(-\infty, 0] \cup [2, +\infty)$

(e) $[0, \frac{\pi}{2}]$

(f) $(\frac{\pi}{2}, \pi]$

(g) $[0, 1]$

(h) $[-\frac{\pi}{4}, \frac{\pi}{4}]$

(i) $[0, \frac{2\pi}{3}]$

5. $A = [0 + 2n\pi, \frac{\pi}{4} + 2n\pi]$ ou $A = [\frac{\pi}{4} + 2n\pi, \frac{\pi}{2} + 2n\pi]$ ou $A = [\frac{\pi}{2} + 2n\pi, \frac{3\pi}{4} + 2n\pi]$ ou $A = [\frac{3\pi}{4} + 2n\pi, \pi + 2n\pi]$ $(n \in \mathbb{Z})$

6.

7.

8.

9. $\frac{\sqrt{5}}{5}$

10.

11. 1/2

12. 0

13. $(0, \frac{\pi}{4})$

14.

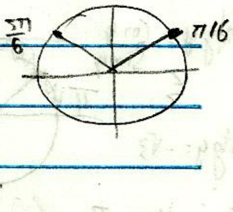
15. $[-1, 1]$

16.

Lista A

1.

a) $y = \arcsin \frac{1}{2} \Leftrightarrow \sin y = \frac{1}{2}$



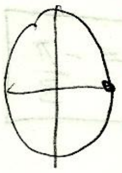
$\Downarrow$

$\text{Im} \arcsin = [-\frac{\pi}{2}, \frac{\pi}{2}]$;

$y = \frac{\pi}{6}, \frac{5\pi}{6}$

Mas $y \in [-\frac{\pi}{2}, \frac{\pi}{2}] \Rightarrow \boxed{y = \frac{\pi}{6}}$

b) $y = \arccos 1 \Leftrightarrow \cos y = 1 \Rightarrow y = 0, 2\pi$



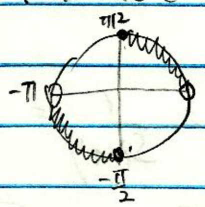
$\text{Im} \arccos = [0, \pi]$;

Mas $y \in [0, \pi] \Rightarrow \boxed{y = 0}$

c) $y = \text{arc csc}(-1) \Leftrightarrow \csc y = -1 \Rightarrow y = \frac{3\pi}{2}$ ou $-\frac{\pi}{2}$

$\text{Im arc csc} = (-\pi, -\frac{\pi}{2}] \cup (0, \frac{\pi}{2}]$;

Mas $y \in (-\pi, -\frac{\pi}{2}] \cup (0, \frac{\pi}{2}]$



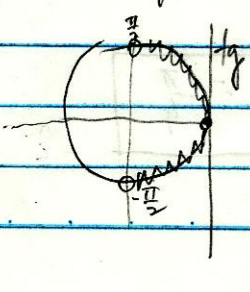
$\csc y = \frac{1}{\sin y}$

$\Rightarrow \boxed{y = -\frac{\pi}{2}}$

d) $y = \text{arc tg}\, 0 \Leftrightarrow \text{tg}\, y = 0 \Rightarrow y = 0, \pi$

$\text{Im arc tg} = (-\frac{\pi}{2}, \frac{\pi}{2})$;

Mas $y \in (-\frac{\pi}{2}, \frac{\pi}{2})$



$\Rightarrow \boxed{y = 0}$

/ /

e) $y = \operatorname{arc\,cotg}(-\sqrt{3}) \Leftrightarrow \operatorname{cotg} y = -\sqrt{3} \Rightarrow y = \frac{-\pi}{6}, \frac{5\pi}{6}$

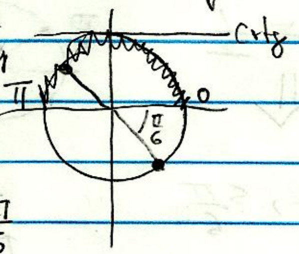
$\operatorname{Im} \operatorname{arc\,cotg} = (0, \pi)$; Mas $y \in (0,\pi) \Rightarrow \boxed{y = \frac{5\pi}{6}}$

$\operatorname{cotg} y = \frac{\cos y}{\sin y}$

$\Leftrightarrow$

$\operatorname{cotg} y = -\sqrt{3}$

$\therefore\ y = \frac{-\pi}{6}$

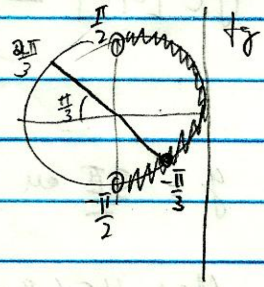


f) $y = \operatorname{arc\,tg}(-\sqrt{3}) \Leftrightarrow \operatorname{tg} y = -\sqrt{3} \Rightarrow y = \frac{-\pi}{3}, \frac{2\pi}{3}$

$\operatorname{Im} \operatorname{arc\,tg} = (-\frac{\pi}{2}, \frac{\pi}{2})$; Mas $y \in (-\frac{\pi}{2}, \frac{\pi}{2}) \Rightarrow \boxed{y = -\frac{\pi}{3}}$

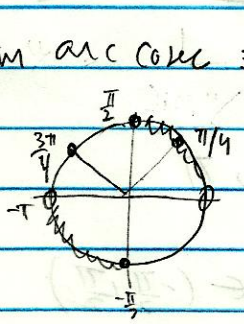
$\operatorname{tg} y = -\sqrt{3}$

$y = \frac{\pi}{3}$



g) $y = \operatorname{arc\,csc}\sqrt{2} \Leftrightarrow \csc y = \sqrt{2}$; $\operatorname{cosec} y = \frac{1}{\sin y} = \sqrt{2}$

$\operatorname{Im} \operatorname{arc\,cosec} = (-\pi, -\frac{\pi}{2}] \cup (0, \frac{\pi}{2}]$; $\sin y = \frac{\sqrt{2}}{2}$



$\Rightarrow \left\} y = \frac{\pi}{4}, \frac{3\pi}{4}\right.$

Mas $y \in (-\pi, -\frac{\pi}{2}) \cup (0, \frac{\pi}{2}]$

$\Rightarrow \boxed{y = \frac{\pi}{4}}$

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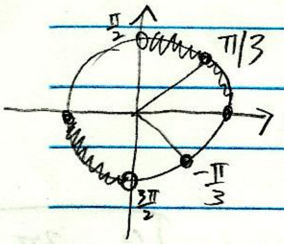
h) $y = \text{arcsec } 2 \Leftrightarrow \sec y = 2 \Leftrightarrow \sec y = \dfrac{1}{\cos y} = 2$

$\text{Im sec} = [0, \frac{\pi}{2}) \cup [\pi, \frac{3\pi}{2})$; $\therefore \cos y = \frac{1}{2}$



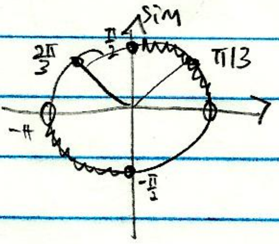
$\Rightarrow y = -\frac{\pi}{3}, \frac{\pi}{3}$

Mas $y \in [0, \frac{\pi}{2}) \cup [\pi, \frac{3\pi}{2})$

$\Rightarrow \boxed{y = \frac{\pi}{3}}$

i) $y = \text{arccosec } \dfrac{2\sqrt{3}}{3} \rightarrow \text{cosec } y = \dfrac{2\sqrt{3}}{3}$

$\text{Im arccosec} = (-\pi, -\frac{\pi}{2}] \cup (0, \frac{\pi}{2}]$; $\sin y = \dfrac{3}{2\sqrt{3}} = \dfrac{\sqrt{3}}{2}$



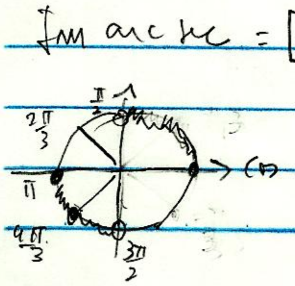
$\Rightarrow y = \frac{\pi}{3}, \frac{2\pi}{3}$

Mas $y \in (-\pi, -\frac{\pi}{2}] \cup (0, \frac{\pi}{2}]$

$\Rightarrow \boxed{y = \frac{\pi}{3}}$

j) $y = \text{arcsec}(-2) \Leftrightarrow \sec y = -2 \Leftrightarrow \dfrac{1}{\cos y} = -2$

$\text{Im arcsec} = [0, \frac{\pi}{2}) \cup [\pi, \frac{3\pi}{2})$; $\cos y = -\frac{1}{2}$



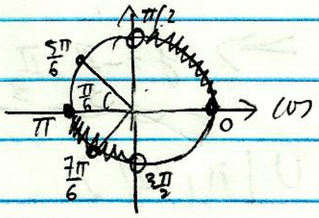
$\Rightarrow y = +\frac{2\pi}{3}, \frac{4\pi}{3}$

Mas

$y \in [0, \frac{\pi}{2}) \cup [\pi, \frac{3\pi}{2}) \Rightarrow \boxed{y = \frac{4\pi}{3}}$

k) $y = \text{arc sec}\left(-\frac{2\sqrt{3}}{3}\right) \Leftrightarrow \sec y = -\frac{2\sqrt{3}}{3}$

$\text{Im arc sec} = [0, \frac{\pi}{2}) \cup [\pi, \frac{3\pi}{2})$; $\cos y = -\frac{3}{2\sqrt{3}} = -\frac{\sqrt{3}}{2}$

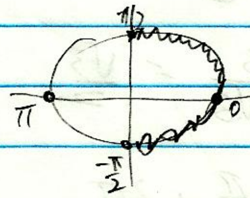


$\Rightarrow y = \frac{5\pi}{6}, \frac{7\pi}{6}$

Mas, $y \in [0, \frac{\pi}{2}) \cup [\pi, \frac{3\pi}{2}) \Rightarrow \boxed{y = \frac{7\pi}{6}}$

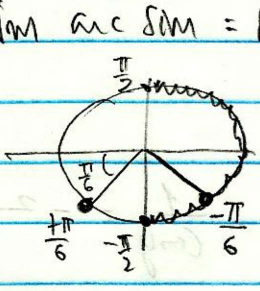
l) $y = \text{arc sin}\, 0 \Leftrightarrow \sin y = 0 \Rightarrow y = 0, \pi$

$\text{Im arc sin} = [-\frac{\pi}{2}, \frac{\pi}{2}]$; $y \in [-\frac{\pi}{2}, \frac{\pi}{2}] \Rightarrow \boxed{y = 0}$



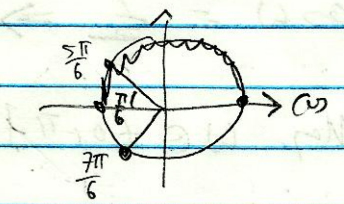
m) $y = \text{arc sin}\, -\frac{1}{2} \Leftrightarrow \sin y = -\frac{1}{2} \Rightarrow y = -\frac{\pi}{6}, \frac{7\pi}{6}$

$\text{Im arc sin} = [-\frac{\pi}{2}, \frac{\pi}{2}]$ $\quad y \in [-\frac{\pi}{2}, \frac{\pi}{2}] \Rightarrow \boxed{y = -\frac{\pi}{6}}$



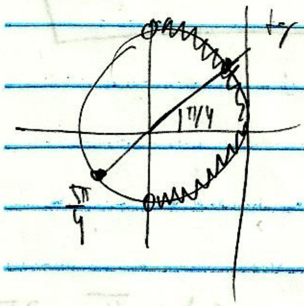
n) $y = \arccos \frac{-\sqrt{3}}{2} \Leftrightarrow \cos y = \frac{-\sqrt{3}}{2} \Rightarrow y = \frac{5\pi}{6}, \frac{7\pi}{6}$

Im arc cos = $[0, \pi]$; Mas $y \in [0, \pi] \Rightarrow \boxed{y = \frac{5\pi}{6}}$



o) $y = \text{arc tg } 1 \Leftrightarrow \text{tg } y = 1 \Rightarrow y = \frac{\pi}{4}, \frac{5\pi}{4}$

Im arc tg = $(-\frac{\pi}{2}, \frac{\pi}{2})$; Mas $y \in (-\frac{\pi}{2}, \frac{\pi}{2}) \Rightarrow \boxed{y = \frac{\pi}{4}}$



2)

a) $y = \sin(\arccos \frac{1}{2})$

Seja, $\omega = \arccos \frac{1}{2} \Leftrightarrow \cos\omega = \frac{1}{2} \Rightarrow \omega = -\frac{\pi}{3}, \frac{\pi}{3}$

$\operatorname{Im} \arccos = [0, \pi]$; Mas $\omega \in [0, \pi]$

$\Rightarrow \omega = \frac{\pi}{3}$

$\therefore$

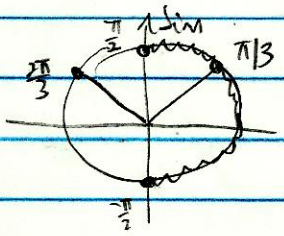
$y = \sin(\underbrace{\arccos \tfrac{1}{2}}_{\omega}) = \sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}$

$\therefore$ $\boxed{\sin(\arccos \tfrac{1}{2}) = \tfrac{\sqrt{3}}{2}}$

b) $tg(\arcsin \frac{\sqrt{3}}{2})$

$\omega = \arcsin \frac{\sqrt{3}}{2} \Leftrightarrow \sin\omega = \frac{\sqrt{3}}{2} \Rightarrow \omega = \frac{\pi}{3}, \frac{2\pi}{3}$

$\operatorname{Im} \arcsin = [-\frac{\pi}{2}, \frac{\pi}{2}]$; Mas $\omega \in [-\frac{\pi}{2}, \frac{\pi}{2}] \Rightarrow \omega = \frac{\pi}{3}$



$\therefore$

$tg(\underbrace{\arcsin \tfrac{\sqrt{3}}{2}}_{\omega}) = tg \frac{\pi}{3} = \sqrt{3}$

$\therefore$ $\boxed{tg(\arcsin \tfrac{\sqrt{3}}{2}) = \sqrt{3}}$

c) $\sec\left(\arccos\frac{\sqrt{3}}{2}\right)$

$\omega = \arccos\frac{\sqrt{3}}{2} \Leftrightarrow \cos\omega = \frac{\sqrt{3}}{2} \Rightarrow \omega = -\frac{\pi}{6}, \frac{\pi}{6}$

$\text{Im}\ \arccos = [0,\pi]$; Mas $\omega \in [0,\pi] \Rightarrow \omega = \frac{\pi}{6}$

$\therefore$

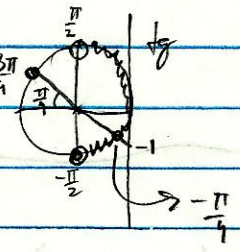
$\sec(\underbrace{\arccos\frac{\sqrt{3}}{2}}_{\omega}) = \sec\frac{\pi}{6} = \frac{1}{\cos\frac{\pi}{6}} = \frac{1}{\frac{\sqrt{3}}{2}} = \frac{2}{\sqrt{3}}$

$\boxed{\sec\left(\arccos\frac{\sqrt{3}}{2}\right) = \frac{2}{\sqrt{3}}}$

d) $\text{cossec}(\text{arc tg}(-1))$

$\omega = \text{arc tg}(-1) \Leftrightarrow \text{tg}\,\omega = -1 \Rightarrow \omega = -\frac{\pi}{4}, \frac{3\pi}{4}$

$\text{Im}\ \text{arctg} = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$



$\omega \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
$\Downarrow$
$\omega = -\frac{\pi}{4}$

$\therefore$

$\text{cossec}(\underbrace{\text{arc tg}(-1)}_{\omega}) = \text{cossec}\left(-\frac{\pi}{4}\right) = \frac{1}{\sin\left(-\frac{\pi}{4}\right)}$

$= \frac{1}{-\frac{\sqrt{2}}{2}}$

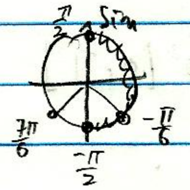
$= -\frac{2}{\sqrt{2}} = -\sqrt{2}$

$\therefore$ $\boxed{\text{cossec}(\text{arc tg}(-1)) = -\sqrt{2}}$

e) $\sin(\arcsin(-\frac{1}{2}))$

$\omega = \arcsin(-\frac{1}{2}) \Longrightarrow \sin\omega = -\frac{1}{2} \Rightarrow \omega = -\frac{\pi}{6}, \frac{7\pi}{6}$

$\text{Im arc sin} = [-\frac{\pi}{2}, \frac{\pi}{2}]$



$\omega \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

$\Downarrow$

$\omega = -\frac{\pi}{6}$

$\therefore$

$\sin(\underbrace{\arcsin(-\tfrac{1}{2})}_{\omega}) = \sin(-\frac{\pi}{6}) = -\sin\frac{\pi}{6} = -\frac{1}{2}$

$\therefore \boxed{\sin(\arcsin(-\tfrac{1}{2})) = -\tfrac{1}{2}}$

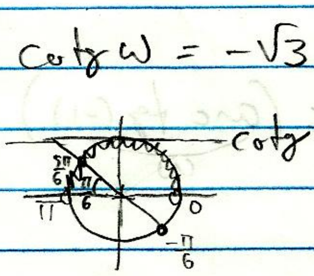
Obs.: Tal resultado já era esperado pois

$\sin(\arcsin x) = x \Longrightarrow \sin(\arcsin -\frac{1}{2}) = -\frac{1}{2}$

f) $\operatorname{cosec}(\operatorname{arc\,cotg}(-\sqrt{3}))$

$\omega = \operatorname{arc\,cotg}(-\sqrt{3})) \Leftrightarrow \operatorname{cotg}\omega = -\sqrt{3} \Longrightarrow \omega = -\frac{\pi}{6}, \frac{5\pi}{6}$

$\text{Im cotg} = (0, \pi)$



$\omega \in (0, \pi)$

$\Downarrow$

$\omega = \frac{5\pi}{6}$

$\therefore$

150°

$\operatorname{cosec}(\underbrace{\operatorname{arc\,cotg}(-\sqrt{3})}_{\omega}) = \operatorname{cosec}(\frac{5\pi}{6}) = \frac{1}{\sin\frac{5\pi}{6}} = \frac{1}{\frac{1}{2}} = 2$

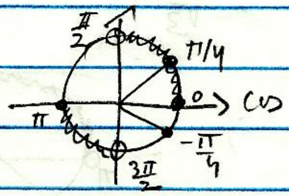
$\therefore \boxed{\operatorname{cosec}(\operatorname{arc\,cotg}(-\sqrt{3})) = \cancel{1/2}}$ 2

tilibra

g) $\text{cosec}(\text{arc sec}\sqrt{2})$

$\omega = \text{arc sec}\sqrt{2} \iff \sec\omega = \sqrt{2} \iff \cos\omega = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$

$\text{Im sec} = [0, \frac{\pi}{2}) \cup [\pi, \frac{3\pi}{2})$



$\Downarrow$

$\omega = -\frac{\pi}{4}, \frac{\pi}{4}$

$\omega \in [0, \frac{\pi}{2}) \cup [\pi, \frac{3\pi}{2})$

$\Longrightarrow \omega = \frac{\pi}{4}$

$\therefore$

$\text{cosec}(\underbrace{\text{arc sec}\sqrt{2}}_{\omega}) = \text{cosec}\frac{\pi}{4} = \frac{1}{\sin\frac{\pi}{4}} = \frac{1}{\frac{\sqrt{2}}{2}}$

$= \frac{2}{\sqrt{2}} = \sqrt{2}$

$\therefore$

$$\boxed{\text{cosec}(\text{arc sec}\sqrt{2}) = \sqrt{2}}$$

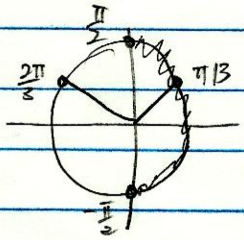
h) $\text{arc sin}(\cos\frac{\pi}{6})$

$y = \text{arc sin}(\cos\frac{\pi}{6}) = \text{arc sin}(\frac{\sqrt{3}}{2}) = y$

$\text{Im arc sin} = [-\frac{\pi}{2}, \frac{\pi}{2}]$

$\iff$

$\sin y = \frac{\sqrt{3}}{2} \Longrightarrow y = \frac{\pi}{3}, \frac{2\pi}{3}$



Mas $y \in [-\frac{\pi}{2}, \frac{\pi}{2}] \Longrightarrow y = \frac{\pi}{3}$

$\therefore$

$$\boxed{\text{arc sin}(\cos\frac{\pi}{6}) = \frac{\pi}{3}}$$

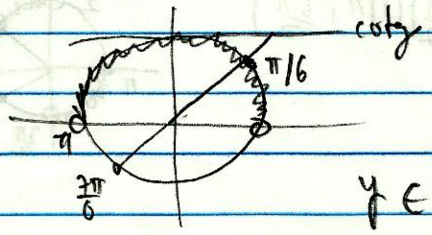
i) arc cotg $\left(tg \frac{\pi}{3}\right)$

$$y = \text{arc cotg}\left(\underbrace{tg\frac{\pi}{3}}_{\sqrt{3}}\right) = \text{arc cotg}\,\sqrt{3} \implies \text{cotg}\, y = \sqrt{3}$$

$$\Downarrow$$

$$y = \frac{\pi}{6}, \frac{7\pi}{6}$$

$\text{Im cotg} = (0, \pi)$;



$$y \in (0, \pi) \implies y = \frac{\pi}{6}$$

$$\therefore \boxed{\text{arc cotg}\left(tg\frac{\pi}{3}\right) = \frac{\pi}{6}}$$

j) arc tan (tan 0)

$$\text{arc tan}(\tan 0) = 0$$

$$f^{-1}(f(0)) = 0$$

$$\therefore \boxed{\text{arc tan}(\tan 0) = 0}$$

3.

a) $f(x) = \dfrac{\cot 2x}{\sin \frac{x}{3}}$

$\cot 2x$: $2x \neq n\pi,\ n \in \mathbb{Z}$

$\therefore\ x \neq \dfrac{n\pi}{2}$ $\underline{\underline{(1)}}$

$\sin \frac{x}{3} \neq 0 \therefore$ $\frac{x}{3} \neq m\pi,\ m \in \mathbb{Z}$

$x \neq 3m\pi,\ m \in \mathbb{Z}$ $\underline{\underline{(2)}}$

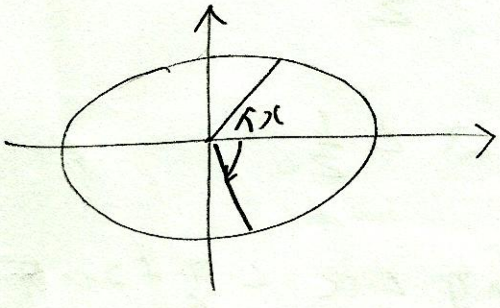
De (1) e (2) : $x \neq n\pi$

$\therefore$ // $\text{Dom } f = \mathbb{R} - \{ \frac{n\pi}{2} : n \in \mathbb{Z} \}$ //

---

b) $f(x) = \sqrt{\cos x}$

$\cos x \geq 0 \implies \begin{cases} x \in [-\frac{\pi}{2} + 2n\pi, \frac{\pi}{2} + 2n\pi] \\ n \in \mathbb{Z} \end{cases}$



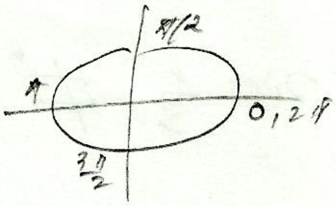
// $\text{Dom } f = \bigcup_{n \in \mathbb{Z}} [-\frac{\pi}{2} + 2n\pi, \frac{\pi}{2} + 2n\pi]$ //

c) $f(x) = (\sin x - 2\sin^2 x)^{-3/4}$

$= \dfrac{1}{\sqrt[4]{(\sin x - 2\sin^2 x)^3}}$

$\sin x - 2\sin^2 x > 0$

$\sin x\,(1 - 2\sin x) > 0$



Seja $y = \sin x$, daí devemos ter

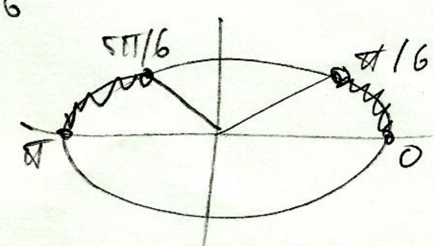
$y(1-2y) > 0$

$- \; -$	0	$+ \; +$			y
$+ \; +$		$+$	$\frac{1}{2}$	$- \; -$	$1-2y$
$-$	0	$+$	$\frac{1}{2}$	$-$	$y(1-2y)$

$y(1-2y) > 0 \Rightarrow 0 < y < \frac{1}{2}$

$\therefore \; 0 < \sin x < \frac{1}{2}$

$\pi - \frac{\pi}{6} = \frac{5\pi}{6}$



$\therefore \; 0 + 2n\pi < x < \frac{\pi}{6} + 2n\pi$

ou

$\frac{5\pi}{6} + 2n\pi < x < \pi + 2n\pi$

$\therefore$ // $\mathrm{Dom}\, f = \mathbb{R} - \bigcup_{n \in \mathbb{Z}} \left\{ \left(2n\pi, \frac{\pi}{6} + 2n\pi\right) \cup \left(\frac{5\pi}{6} + 2n\pi, (2n+1)\pi\right) \right\}$ //

d. $f(x) = \arccos(3-x)$

$3-x \in [-1,1]$

$\therefore \quad -1 \leq 3-x \leq 1 \quad \therefore \quad -4 \leq -x \leq -2$

$4 \geq x \geq 2$

$\therefore$ // $\mathrm{Dom}\, f = [2,4]$ //

---

e. $f(x) = \arcsin\left(\frac{1}{2}x-1\right) + \arccos\left(1-\frac{1}{2}x\right)$

$\arcsin\left(\frac{1}{2}x-1\right)$: $\quad \frac{1}{2}x-1 \in [-1,1]$

$\therefore \quad -1 \leq \frac{1}{2}x-1 \leq 1$

$0 \leq \frac{1}{2}x \leq 2$

$0 \leq x \leq 4 \quad (*)$

$\arccos\left(1-\frac{1}{2}x\right)$: $\quad 1-\frac{1}{2}x \in [-1,1]$

$\therefore \quad -1 \leq 1-\frac{1}{2}x \leq 1$

$-2 \leq -\frac{1}{2}x \leq 0$

$2 \geq \frac{1}{2}x \geq 0$

$4 \geq x \geq 0$ (Coincide con (*))

$\therefore$ // $\mathrm{Dom}\, f = [0,4]$ //

f. $f(x) = 3 \arcsin \sqrt{\frac{3x-1}{2}}$

$\rightarrow \sqrt{\frac{3x-1}{2}}$: $\frac{3x-1}{2} \geq 0$ (*)

$\rightarrow \arcsin \sqrt{\frac{3x-1}{2}}$: $\sqrt{\frac{3x-1}{2}} \in [-1,1]$ (**)

$\frac{3x-1}{2} \geq 0$ $\therefore$ $3x-1 \geq 0$

$x \geq \frac{1}{3}$ (3*)

$\sqrt{\frac{3x-1}{2}} \in [-1,1]$ $\therefore$ $0 \leq \sqrt{\frac{3x-1}{2}} \leq 1$

$\therefore$ $0 \leq \frac{3x-1}{2} \leq 1$

$\therefore$ $0 \leq 3x-1 \leq 2$

$1 \leq 3x \leq 3$

$\frac{1}{3} \leq x \leq 1$ (4*)

De (3*) e (4*) : $\frac{1}{3} \leq x \leq 1$

$\therefore$ // $Dom\ f = [\frac{1}{3}, 1]$ //

g. $f(x) = \arccos \frac{1}{x-1}$

$\frac{1}{x-1}$: $x \neq 1$ (*)

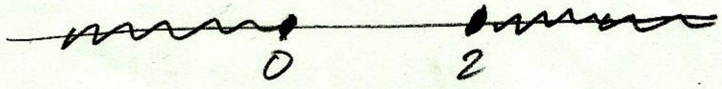
$\arccos \frac{1}{x-1}$: $\frac{1}{x-1} \in [-1,1]$

$\therefore \quad -1 \leq \frac{1}{x-1} \leq 1$

$-1 \leq \frac{1}{x-1} \leq 0$ ou $0 \leq \frac{1}{x-1} \leq 1$

$\therefore \quad -1 \geq x-1$ ou $x-1 \geq 1$

$0 \geq x$ ou $x \geq 2$



(**)

h. (*) e (**) : $x \leq 0$ ou $x \geq 2$

$\therefore$ $\boxed{\text{Dom } f = (-\infty, 0] \cup [2, +\infty)}$

4.

a) $f(x) = 1 - 2|\cos x|$; $\operatorname{Dom} f = \mathbb{R}$

$y = 1 - 2|\cos x|$

$\therefore \quad |\cos x| = \frac{1-y}{2}$

Mas $|\cos x| \in [0,1]$

$\therefore \quad 0 \leq \frac{1-y}{2} \leq 1$

$\therefore \quad 0 \leq 1-y \leq 2$

$\therefore \quad -1 \leq -y \leq 1$

$\therefore \quad 1 \geq y \geq -1$

// $\operatorname{Im} f = [-1,1]$ //

b. $f(x) = \sin x + \sin\left(x + \frac{\pi}{3}\right)$; $\text{Dom} f = \mathbb{R}$

$$y = \sin x + \sin\left(x + \frac{\pi}{3}\right) \quad (*)$$

$$= \sin x + \sin x \cos\frac{\pi}{3} + \sin\frac{\pi}{3}\cos x$$

$$= \sin x + \sin x \, \frac{1}{2} + \frac{\sqrt{3}}{2}\cos x$$

$$= \frac{3}{2}\sin x + \frac{\sqrt{3}}{2}\cos x$$

$$= \frac{3}{2}\sin x + \frac{\sqrt{3}}{2}\left(\pm\sqrt{1-\sin^2 x}\right)$$

$$= \frac{3}{2}\sin x \pm \frac{\sqrt{3}}{2}\sqrt{1-\sin^2 x}$$

$\therefore$

$$y - \frac{3}{2}\sin x = \pm\frac{\sqrt{3}}{2}\sqrt{1-\sin^2 x}$$

$$\left(y - \frac{3}{2}\sin x\right)^2 = \left(\pm\frac{\sqrt{3}}{2}\sqrt{1-\sin^2 x}\right)^2$$

$$y^2 - 3y\sin x + \frac{9}{4}\sin^2 x = \frac{3}{4}(1-\sin^2 x)$$

$$y^2 - 3y\sin x + \frac{9}{4}\sin^2 x - \frac{3}{4} + \frac{3}{4}\sin^2 x = 0$$

$$3\sin^2 x - 3y\sin x + \left(y^2 - \frac{3}{4}\right) = 0$$

Seja $z = \sin x$

Daí devemos ter : $3z^2 - 3yz + \left(y^2 - \frac{3}{4}\right) = 0 \quad (**)$

I.e. $(*)$ é equivalente à $(**)$, o que nos leva a termos soluções de $(**)$. Logo devemos ter

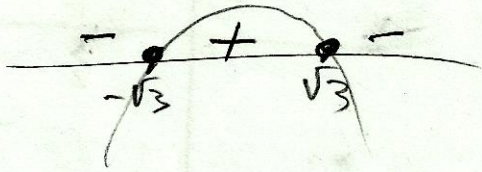
$$\Delta = (-3y)^2 - 4 \cdot 3 \cdot \left(y^2 - \frac{3}{4}\right) \geq 0$$

$$\therefore \quad 9y^2 - 12\left(y^2 - \frac{3}{4}\right) \geq 0$$

$$9y^2 - 12y^2 + 9 \geq 0$$

$$-3y^2 + 9 \geq 0$$

$$-y^2 + 3 \geq 0$$



$$\therefore \quad -\sqrt{3} \leq y \leq \sqrt{3}$$

$$\therefore \quad // \operatorname{Im} f = [-\sqrt{3}, \sqrt{3}] //$$

---

c. $f(x) = \sin^4 x + \cos^4 x$

$$y = \sin^4 x + \cos^4 x = \sin^4 x + (\cos^2 x)^2 \geq 0 \quad (*)$$

$$= \sin^4 x + (1 - \sin^2 x)^2$$

$$= \sin^4 x + 1 - 2\sin^2 x + \sin^4 x$$

$$y = 2\sin^4 x - 2\sin^2 x + 1$$

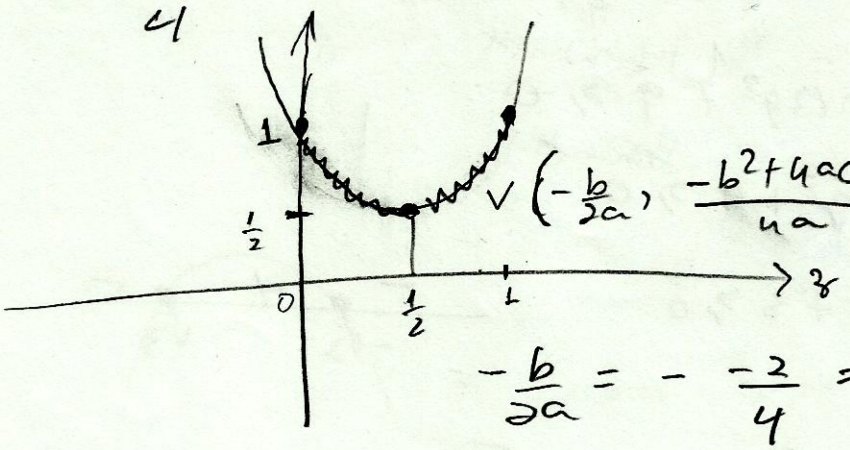
Seja $z = \sin^2 x \quad \therefore \quad 0 \leq z \leq 1 \quad (**)$

Daí temos $\quad y = 2z^2 - 2z + 1 \quad (***)$

Analisemos agora o gráfico de $\underline{2z^2 - 2z + 1}$ na faixa dada em $(**)$ i.e.

$$2z^2 - 2z + 1 = 0$$

$$z = \frac{2 \pm \sqrt{4-8}}{4} \Longrightarrow \nexists\, z \in \mathbb{R}$$



$$V\left(-\frac{b}{2a}, \frac{-b^2+4ac}{4a}\right)$$

$$-\frac{b}{2a} = -\frac{-2}{4} = \frac{1}{2}$$

$$\frac{-b^2+4ac}{4a} = \frac{-4+4\cdot 2\cdot 1}{8} = \frac{4}{8} = \frac{1}{2}$$

$$\therefore \quad \underline{\underline{V = \left(\frac{1}{2}, \frac{1}{2}\right)}}$$

Mas $0 \leq z \leq 1 \Longrightarrow \frac{1}{2} \leq y \leq 1$ (***)

De (*) e (***) :

$$\frac{1}{2} \leq y \leq 1$$

$$\therefore \quad \boxed{\operatorname{Im} f = \left[\frac{1}{2}, 1\right]}$$

d. $f(x) = \dfrac{1+\sin x}{\sin x} \Longrightarrow x \neq n\pi,\ n \in \mathbb{Z}$

Seja

$$y = \frac{1+\sin x}{\sin x}$$

$$\therefore \quad y \sin x = 1 + \sin x$$

$$y \sin x - \sin x = 1$$

$$\sin x\,(y-1) = 1$$

$$\sin x = \frac{1}{y-1}$$

Mas

$$-1 \le \sin x \le 1 \Longrightarrow -1 \le \frac{1}{y-1} \le 1$$

$$\therefore \quad -1 \le \frac{1}{y-1} \le 0 \quad \text{ou} \quad 0 \le \frac{1}{y-1} \le 1$$

Mas

$-1 \le \dfrac{1}{y-1} \le 0$ ou $0 \le \dfrac{1}{y-1} \le 1$

$\therefore \quad -1 \ge y-1$ | $\therefore \quad y-1 \ge 1$

$\therefore \quad 0 \ge y$ (*) | $y \ge 2$ (**)

$$\mathrm{Im}\, f = (-\infty, 0] \cup [2, +\infty)$$

e. $f(x) = \arccos|x|$

Temos : $\text{Dom } \arccos = [-1,1]$

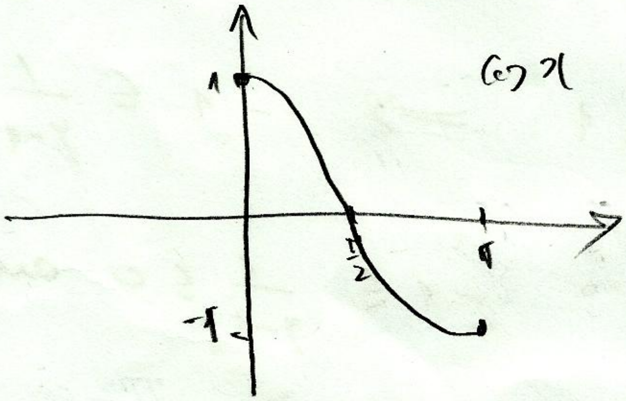
$\text{Im } \arccos = [0,\pi]$

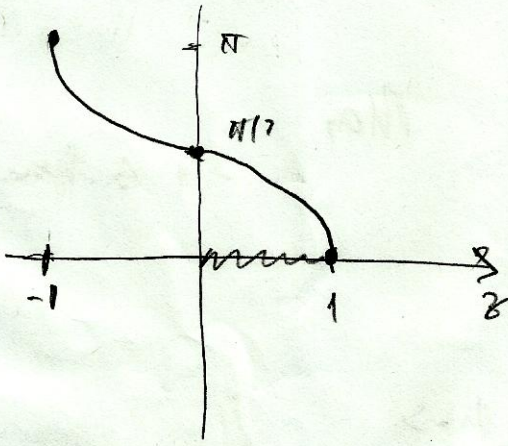
Seja $z = |x|$

$\therefore$ $z \geq 0$

Daí $y = \arccos|x| = \arccos z$ com $0 \leq z \leq 1$ (exclui-se $-1 \leq z \leq 0$ para $z \geq 0$)

Mas





Quando $0 \leq z \leq 1$

temos

$0 \leq y = \arccos z \leq \frac{\pi}{2}$

$\therefore$ $\boxed{\text{Im } f = [0, \frac{\pi}{2}]}$

f.

$$f(x) = \pi - |\arctan x|$$

Seja $z = \arctan x$

Temos: $\begin{cases} \text{Dom } \arctan = \mathbb{R} \\ \text{Im } \arctan = (-\frac{\pi}{2}, \frac{\pi}{2}) \end{cases}$

Daí $|z| \in [0, \frac{\pi}{2})$ $\therefore$ $0 \leq |z| < \frac{\pi}{2}$ (*)

Temos então que

$$y = \pi - |\arctan x|$$

$$\therefore \quad \underbrace{|\arctan x|}_{|z|} = \pi - y$$

$$|z| = \pi - y$$

De (*): $0 \leq |z| < \frac{\pi}{2} \Rightarrow 0 \leq \pi - y < \frac{\pi}{2}$

$$\therefore \quad -\pi \leq -y < \frac{\pi}{2} - \pi$$

$$-\pi \leq -y < -\frac{\pi}{2}$$

$$\therefore \quad \pi \geq y > \frac{\pi}{2}$$

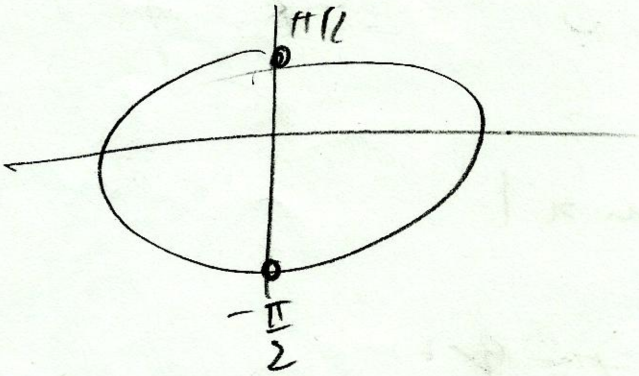
$$\therefore \quad \text{Im } f = (\tfrac{\pi}{2}, \pi]$$

9. $f(x) = \cos \operatorname{arc\,sen} x$

Seja $z = \operatorname{arc\,sen} x$

$\therefore$ $z \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

Daí $y = \cos \operatorname{arc\,sen} x = \cos z$



Mas, quando

$-\frac{\pi}{2} \leq z \leq \frac{\pi}{2}$

temos:

$0 \leq \cos z \leq 1$

e $\therefore$ $y \in [0, 1]$

$\therefore \| \operatorname{Im} f = [0, 1] \|$

h.. $f(x) = \arctan \frac{2x}{x^2+1}$

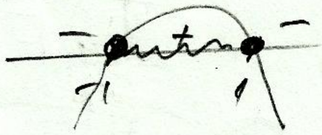
Seja $z = \frac{2x}{x^2+1}$; $z \in \mathbb{R}$ (*)

$\therefore$ $zx^2 + z = 2x$

$zx^2 - 2x + z = 0$

$\Delta = 4 - 4z^2 \geq 0$

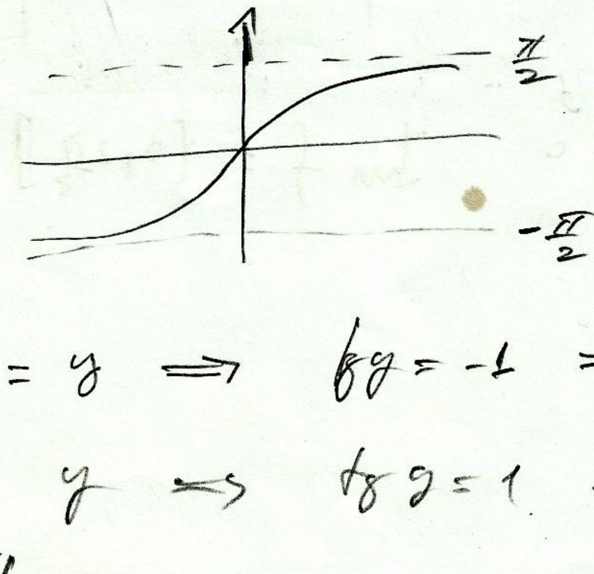
$\therefore$ $1 - z^2 \geq 0$



$\therefore$ $-1 \leq z \leq 1$ (*1)

De (*) e (*1) : $-1 \leq z \leq 1$.

Mas arctg x é função injetora,



$\operatorname{arctg} -1 = y \Rightarrow \operatorname{tg} y = -1 \Rightarrow y = -\frac{\pi}{4}$

$\operatorname{arctg} 1 = y \Rightarrow \operatorname{tg} y = 1 \Rightarrow y = \frac{\pi}{4}$

$\therefore$ $\| \operatorname{Im} f = [-\frac{\pi}{4}, \frac{\pi}{4}] \|$

i. $f(x) = \arccos \frac{3x-1}{2}$; $x \in [0,1]$

$z = \frac{3x-1}{2}$ ∴ $2z = 3x - 1$

∴ $x = \frac{2z+1}{3}$

$x \in [0,1] \Rightarrow 0 \leq \frac{2z+1}{3} \leq 1$

$3 \cdot 0 \leq 2z+1 \leq 3$

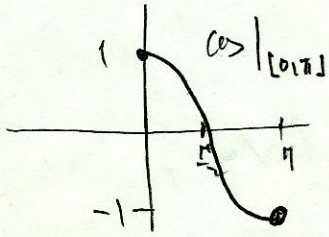
$0 \leq 2z+1 \leq 3$

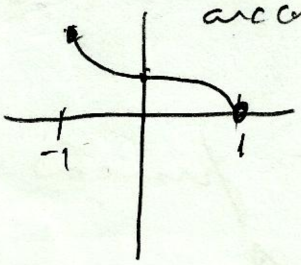
$-1 \leq 2z \leq 2$

$-\frac{1}{2} \leq z \leq 1$ arccos função decrescente

∴ arccos $\begin{cases} \underline{\arccos 1} \leq \arccos z \leq \\ \leq \underline{\arccos -\frac{1}{2}} \end{cases}$

$y = \arccos z$





∴ $0 \leq \arccos z \leq \frac{2\pi}{3}$

∴ $\text{Im } f = \left[0, \frac{2\pi}{3}\right]$

3. $f : A \to [0,1]$

$x \to f(x) = \sin^2 2x$

Seja inicialmente $A \subset [0, 2\pi]$.

Sendo $f(x) = \sin^2 2x \geq 0$ podemos restringir $2x$ ao intervalo $[0, \frac{\pi}{2}]$

i.e. $0 \leq 2x \leq \frac{\pi}{2}$

$\therefore 0 \leq x \leq \frac{\pi}{4}$

→ Assim uma possibilidade é tomar

$$// A = [0, \frac{\pi}{4}] //$$

→ outra possibilidade seria tomar:

$2x \in [\frac{\pi}{2}, \pi] \quad \therefore // x \in [\frac{\pi}{4}, \frac{\pi}{2}] //$

→ outra possibilidade seria tomar

$2x \in [\pi, \frac{3\pi}{2}] \quad \therefore // x \in [\frac{\pi}{2}, \frac{3\pi}{4}] //$

→ outra possibilidade seria tomar

$2x \in [\frac{3\pi}{2}, 2\pi] \quad \therefore // x \in [\frac{3\pi}{4}, \pi] //$

6) a) $\sec(\operatorname{arctg} x)$:

Seja $y = \operatorname{arctg} x \Leftrightarrow \operatorname{tg} y = x$

Então,

$$\sec(\operatorname{arctg} x) = \sec y = \pm\sqrt{1+\operatorname{tg}^2 y}$$

$\therefore$

$$\sec(\operatorname{arctg} x) = \pm\sqrt{1+x^2} \quad (*)$$

Mas,

$\operatorname{Im}\ y = \operatorname{arctg} x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$\sec y \geq 0 \quad , \quad y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

Logo $\sec \operatorname{arctg} x \geq 0$

o que nos leva a tomar o sinal positivo em $(*)$:

i.e.,

$$\boxed{\sec(\operatorname{arctg} x) = \sqrt{1+x^2}}$$

b) $\sin(\text{arc cosec } x)$

Seja

$y = \text{arc cosec } x \implies \text{cosec } y = x$

então

$\sin(\text{arc cosec } x) = \sin y = \frac{1}{\text{cosec } y} = \frac{1}{x}$

$\therefore$

$$\boxed{\sin(\text{arc cosec } x) = \frac{1}{x}}$$

c) $\cos(2 \text{arc sin } x)$

Seja,

$y = 2 \text{arc sin } x \iff \begin{cases} \frac{y}{2} = \text{arc sin } x \\ \iff \\ \sin \frac{y}{2} = x \end{cases}$

$\therefore$

$\cos(2 \text{arc sin } x) = \cos y$

$= 1 - 2\sin^2 \frac{y}{2}$

$$\boxed{\cos(2 \text{arc sin } x) = 1 - 2x^2}$$

$\sin^2 y = \frac{1 - \cos 2y}{2}$

$\iff$

$2\sin^2 y = 1 - \cos 2y$

$\iff$

$\cos 2y = 1 - 2\sin^2 y$

$\iff$

$\cos y = 1 - 2\sin^2 \frac{y}{2}$

d) $\sin(2\arcsin x)$

Seja $y = 2\arcsin x \iff \frac{y}{2} = \arcsin x$
$$\iff$$
$$\sin\frac{y}{2} = x$$

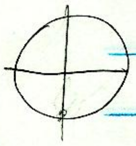
Então,

$$\sin(2\arcsin x) = \underbrace{\sin y}$$
$$= 2\underbrace{\sin\frac{y}{2}}\,\underbrace{\cos\frac{y}{2}}$$
$$= 2\underbrace{x}\left(\pm\sqrt{1-x^2}\right)$$

$\sin y = 2\sin\frac{y}{2}\cos\frac{y}{2}$

$\cos\frac{y}{2} = \pm\sqrt{1-\sin^2\frac{y}{2}}$
$= \pm\sqrt{1-x^2}$

Mas,



$$\begin{cases} -\frac{\pi}{2} \leq \underbrace{\arcsin x} \leq \frac{\pi}{2} \\ -\frac{\pi}{2} \leq \frac{y}{2} \leq \frac{\pi}{2} \end{cases} \Longrightarrow \underline{0 \leq \cos\frac{y}{2} \leq 1}$$

$\Longrightarrow$
$$\underline{\cos\frac{y}{2} = +\sqrt{1-x^2}}$$

Logo,
$$\boxed{\sin(2\arcsin x) = 2x\sqrt{1-x^2}}$$

7)

a) Prove que,

$$\arcsin x + \arcsin y = \arcsin\left(x\sqrt{1-y^2} + y\sqrt{1-x^2}\right)$$

desde que o valor da expressão do lado esquerdo esteja entre $[-\frac{\pi}{2}, \frac{\pi}{2}]$.

<u>Demonstração</u>:

Seja,

$$\begin{cases} a = \arcsin x \Leftrightarrow \sin a = x \\ b = \arcsin y \Leftrightarrow \sin b = y \end{cases}$$

e $a+b \in [-\frac{\pi}{2}, \frac{\pi}{2}]$.

Então, queremos mostrar que se tem

$$a+b = \arcsin\left(x\sqrt{1-y^2} + y\sqrt{1-x^2}\right)$$

$$\Leftrightarrow$$

$$// \sin(a+b) = x\sqrt{1-y^2} + y\sqrt{1-x^2} //$$

Com efeito,

$$\sin(a+b) = \sin a \cos b + \sin b \cos a$$

$$\sin(a+b) = x\sqrt{1-y^2} + y\sqrt{1-x^2} \Longrightarrow$$

$a+b = \arcsin\left(x\sqrt{1-y^2} + y\sqrt{1-x^2}\right)$

$\sim$

$$\boxed{\arcsin x + \arcsin y = \arcsin\left(x\sqrt{1-y^2} + y\sqrt{1-x^2}\right)}$$

---

5) Mostre que,

(*) $\quad \operatorname{arctg} \dfrac{x}{\sqrt{1-x^2}} = \arcsin x \quad , \quad -1 < x < 1$

<u>Solução</u>

Inicialmente lembraremos que

$$\begin{cases} \text{Dom } \operatorname{arctg} x = (-\infty, +\infty) \qquad (-1,1) \\ \text{Im } \operatorname{arctg} x = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \\ \\ \text{Dom } \left(h(x) = \dfrac{x}{\sqrt{1-x^2}}\right) = (-1,1) \\ \\ \text{Im } h(x) = (-\infty\ +\infty) \\ \\ \text{Dom } \arcsin x = [-1,1] \\ \text{Im } \arcsin x = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \end{cases}$$

Daí

$$\operatorname{arctg} \frac{x}{\sqrt{1-x^2}} = \operatorname{arctg} \circ h(x)$$

$$\Downarrow$$

$$\text{Dom } \operatorname{arctg} \circ h(x) = \text{Dom } h(x) = (-1,1)$$

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o que mostra que a identidade ⊛ só faz sentido se definida no intervalo $-1 < x < 1$.

Seja

$$y = \operatorname{arc\,tg} \frac{x}{\sqrt{1-x^2}} \Leftrightarrow \operatorname{tg} y = \frac{x}{\sqrt{1-x^2}}$$

$$\frac{\operatorname{sen} y}{\cos y} = \frac{x}{\sqrt{1-x^2}}$$

Aqui, $\cos y = \pm\sqrt{1-\operatorname{sen}^2 y}$

nos dá,

$$\frac{\operatorname{sen} y}{\pm\sqrt{1-\operatorname{sen}^2 y}} = \frac{x}{\sqrt{1-x^2}}$$

que só será satisfeita se tomarmos

$\cos y = \sqrt{1-\operatorname{sen}^2 y}$, e $\underline{\operatorname{sen} y = x}$

Daí,

$$\operatorname{sen} y = x \Leftrightarrow \underline{y} = \operatorname{arc\,sen} x$$

$$\boxed{\operatorname{arc\,tg} \frac{x}{\sqrt{1-x^2}} = \operatorname{arc\,sen} x}$$

c) prove que

$$\begin{cases} \operatorname{arctg} x + \operatorname{arctg} y = \operatorname{arctg} \dfrac{x+y}{1-xy} \quad \text{para } xy \neq 1 \\ \text{Com } \operatorname{arctg} x + \operatorname{arctg} y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \end{cases}$$

Solução

$$\left.\begin{aligned} a &= \operatorname{arctg} x \\ b &= \operatorname{arctg} y \end{aligned}\right\} \qquad \begin{aligned} \operatorname{tg} a &= x \\ \operatorname{tg} b &= y \end{aligned}$$

$$\therefore \quad \underbrace{\frac{x+y}{1-xy} = \frac{\operatorname{tg} a + \operatorname{tg} b}{1 - \operatorname{tg} a \operatorname{tg} b} = \operatorname{tg}(a+b)}_{\Leftarrow}$$

$$\operatorname{arc\,tg}\left(\frac{x+y}{1-xy}\right) = a + b$$

$$\Bigg/\!\!\Bigg/ \operatorname{arc\,tg}\left(\frac{x+y}{1-xy}\right) = \operatorname{arctg} x + \operatorname{arctg} y \Bigg/\!\!\Bigg/$$

obs.:

$$\begin{cases} \text{Dom } \operatorname{arc\,tg} = (-\infty, +\infty) \\ \text{Im } \operatorname{arc\,tg} = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \end{cases}$$

d) Sejam a, b, c números satisfazendo

$bc = 1 + a^2$. Mostre que,

$$\operatorname{arctg} \frac{1}{a+b} + \operatorname{arctg} \frac{1}{a+c} = \operatorname{arctg} \frac{1}{a}$$

se $\begin{cases} a+b \neq 0, \quad a+c \neq 0, \quad a \neq 0 \\ \text{e com } \operatorname{arctg} \frac{1}{a+b} + \operatorname{arctg} \frac{1}{a+c} \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \end{cases}$

Solução:

Seja,

$$y = \operatorname{arctg} \frac{1}{a+b} \Leftrightarrow \operatorname{tg} y = \frac{1}{a+b}$$

$$z = \operatorname{arctg} \frac{1}{a+c} \Leftrightarrow \operatorname{tg} z = \frac{1}{a+c}$$

Então

$$\begin{matrix} a+b = \dfrac{1}{\operatorname{tg} y} \Rightarrow \\ a+c = \dfrac{1}{\operatorname{tg} z} \Rightarrow \end{matrix} \begin{cases} b = \operatorname{tg}^{-1} y - a \\ c = \operatorname{tg}^{-1} z - a \end{cases}$$

$\therefore$

$$bc = (\operatorname{tg}^{-1} y - a)(\operatorname{tg}^{-1} z - a)$$

$$bc = \operatorname{tg}^{-1} y \operatorname{tg}^{-1} z - a(\operatorname{tg}^{-1} y + \operatorname{tg}^{-1} z) + a^2$$

$$1 + \cancel{a^2} = \operatorname{tg}^{-1} y \operatorname{tg}^{-1} z - a(\operatorname{tg}^{-1} y + \operatorname{tg}^{-1} z) \cancel{+ a^2}$$

$\therefore$ $$\left\| a = \frac{\operatorname{tg}^{-1} y \operatorname{tg}^{-1} z - 1}{\operatorname{tg}^{-1} y + \operatorname{tg}^{-1} z} \right\| \Longrightarrow$$

$$\frac{1}{a} = \frac{tg^{-1}y + tg^{-1}z}{tg^{-1}y\, tg^{-1}z - 1} = \frac{\frac{1}{tg\,y} + \frac{1}{tg\,z}}{\frac{1}{tg\,y}\frac{1}{tg\,z} - 1}$$

$$\frac{1}{a} = \frac{tg\,z + tg\,y}{1 - tg\,y\, tg\,z}$$

Mas

$$tg\,(y \pm z) = \frac{tg\,y \pm tg\,z}{1 \mp tg\,y\, tg\,z}$$

$$\frac{1}{a} = tg\,(y+z)$$

$$\Leftrightarrow$$

$$arctg\,\frac{1}{a} = y + z \qquad \text{com } y+z \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

$$\boxed{arctg\,\frac{1}{a} = arctg\,\frac{1}{a+b} + arctg\,\frac{1}{a+c}}$$

e) $\arcsin\left(\frac{x}{3}-1\right) = \frac{\pi}{2} - 2\arcsin\sqrt{1-\frac{x}{6}}$

Seja

$$\left\| \; y = \frac{\pi}{2} - 2\arcsin\sqrt{1-\frac{x}{6}} \; \right\| \quad (*)$$

$\Leftrightarrow$

$$\frac{\frac{\pi}{2}-y}{2} = \arcsin\sqrt{1-\frac{x}{6}}$$

$$\frac{\pi}{4} - \frac{y}{2} = \arcsin\sqrt{1-\frac{x}{6}}$$

$\Downarrow$

$$\underbrace{\sin\left(\frac{\pi}{4}-\frac{y}{2}\right)} = \sqrt{1-\frac{x}{6}}$$

$$\sin\frac{\pi}{4}\cos\frac{y}{2} - \sin\frac{y}{2}\cos\frac{\pi}{4} = \sqrt{1-\frac{x}{6}}$$

$$\frac{\sqrt{2}}{2}\cos\frac{y}{2} - \sin\frac{y}{2}\frac{\sqrt{2}}{2} = \sqrt{1-\frac{x}{6}}$$

$$\frac{\sqrt{2}}{2}\left(\cos\frac{y}{2} - \sin\frac{y}{2}\right) = \sqrt{1-\frac{x}{6}}$$

elevando ao quadrado

$$\frac{2}{4}\left(\underbrace{\cos^2\frac{y}{2} + \sin^2\frac{y}{2}} - \underbrace{2\cos\frac{y}{2}\sin\frac{y}{2}}\right) = 1-\frac{x}{6}$$

$$\frac{1}{2}\left(1 - \underbrace{\sin y}\right) = 1-\frac{x}{6}$$

$$1-\sin y = 2-\frac{x}{3}$$

$$-\sin y = 1-\frac{x}{3}$$

$$\sin y = \frac{x}{3}-1 \Longrightarrow \left\| \; y = \arcsin\left(\frac{x}{3}-1\right) \; \right\|$$

$(**)$

/ /

De (*) e (**) obtemos que

$$\boxed{\arcsin\left(\frac{x}{3}-1\right) = \frac{\pi}{2} - 2\arcsin\sqrt{1-\frac{x}{6}}}$$

→

Seja

$$y = 2\arcsin\frac{\sqrt{x}}{\sqrt{6}} - \frac{\pi}{2}$$

$$\Leftrightarrow$$

$$\frac{y+\frac{\pi}{2}}{2} = \arcsin\frac{\sqrt{x}}{\sqrt{6}}$$

$$\Leftrightarrow$$

$$\sin\left(\frac{y}{2}+\frac{\pi}{4}\right) = \frac{\sqrt{x}}{\sqrt{6}}$$

$$\sin\frac{y}{2}\cos\frac{\pi}{4} + \sin\frac{\pi}{4}\cos\frac{y}{2} = \frac{\sqrt{x}}{\sqrt{6}}$$

$$\frac{\sqrt{2}}{2}\left(\sin\frac{y}{2} + \cos\frac{y}{2}\right) = \frac{\sqrt{x}}{\sqrt{6}}$$

$$\frac{1}{2}\cancel{\frac{2}{4}}\left(\underbrace{\sin^2\frac{y}{2} + \cos^2\frac{y}{2}} + \underbrace{2\sin\frac{y}{2}\cos\frac{y}{2}}\right) = \frac{x}{6}$$

$$\frac{1}{\cancel{2}}\left(1 + \underbrace{\sin y}\right) = \frac{x}{\cancel{6}\,3}$$

$$\sin y = \frac{x}{3} - 1$$

$$\Leftrightarrow$$

$$y = \arcsin\left(\frac{x}{3}-1\right)$$

$$\boxed{2\left(\arcsin\frac{\sqrt{x}}{\sqrt{6}}\right) - \frac{\pi}{2} = \arcsin\left(\frac{x}{3}-1\right)}$$

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f) Mostrar que

$\arcsin x + \arccos x = c$, com $-1 \leq x \leq 1$

Seja

$$\begin{cases} y \equiv \arcsin x \Leftrightarrow \sin y = x, \quad y \in [-\frac{\pi}{2}, \frac{\pi}{2}] \quad (1) \\ z = \arccos x \Leftrightarrow \cos z = x, \quad z \in [0, \pi] \quad (2) \end{cases}$$

$\therefore$

$\sin y = \cos z$ (*)

Análise: Devemos encontrar y e z satisfazendo (*).

Seja

i) $y = z + \frac{\pi}{2}$

$\Leftrightarrow$

$\sin y = \sin(z + \frac{\pi}{2}) =$

$= \sin z \cancel{\cos \frac{\pi}{2}}^{0} + \sin \frac{\pi}{2} \cos z$

$= \cos z$ satisfaz (*)

Mas, sendo $y = z + \frac{\pi}{2}$, e $z \in [0, \pi]$ temos que

$z = 0 \Rightarrow y = \frac{\pi}{2}$

$z = \pi \Rightarrow y = \frac{3\pi}{2}$

$\Rightarrow y \in [\frac{\pi}{2}, \frac{3\pi}{2}]$ o que contraria (1).

Logo a escolha $y = z + \frac{\pi}{2}$ não serve

/ /

(ii) $y = \frac{\pi}{2} - z$

$\Leftarrow$

$\sin y = \sin(\frac{\pi}{2} - z)$

$= \sin\frac{\pi}{2}\cos z - \sin z \cos\frac{\pi}{2}^{\,0}$

$= \cos z$ satisfaz (*)

Sendo $y = \frac{\pi}{2} - z$ e $z \in [0, \pi]$ temos que

$z = 0 \Rightarrow y = \frac{\pi}{2}$

$z = \pi \Rightarrow y = \frac{\pi}{2} - \pi = -\frac{\pi}{2}$

$\Rightarrow y \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

OK!

(satisfaz (1))

Isto nos dá então a escolha:

$y = \frac{\pi}{2} - z$

$\therefore$

$y + z = \frac{\pi}{2}$

$\arcsin x + \arccos x = \frac{\pi}{2}$

i.é. $\boxed{c = \frac{\pi}{2}}$

$\therefore$

$$\boxed{\arcsin x + \arccos x = \frac{\pi}{2}}$$

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g) $f(x) = \text{arc tg } x + \text{arc tg } \frac{1}{x}$

Seja

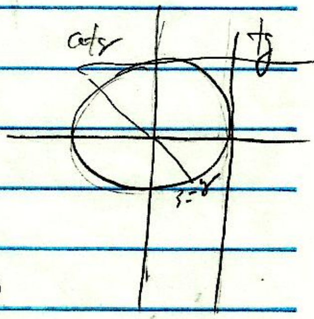
$$\left\{ \begin{array}{l} y = \text{arc tg } x \Leftrightarrow \underline{tg\, y = x} \quad (*) \\ \text{Dom arc tg } x = \mathbb{R} \\ \text{Im arc tg} = (-\frac{\pi}{2}, \frac{\pi}{2}) \Longrightarrow \boxed{y \in (-\frac{\pi}{2}, \frac{\pi}{2})} \quad (1) \end{array} \right.$$

$$\left\{ \begin{array}{l} z = \text{arc tg } \frac{1}{x} \Leftrightarrow tg\, z = \frac{1}{x} \Leftrightarrow \underline{cotg\, z = x} \quad (**) \\ \text{Dom arc tg} = \mathbb{R} \\ \text{Im arc tg} = (-\frac{\pi}{2}, \frac{\pi}{2}) \Longrightarrow \boxed{z \in (-\frac{\pi}{2}, \frac{\pi}{2})} \quad (2) \end{array} \right.$$

De (*) e (**) tem-se:

$$tg\, y = cotg\, z = x$$

i) Seja $x \in (-\infty, 0)$



então

$$tg\, y = x < 0 \overset{(1)}{\Longrightarrow} -\frac{\pi}{2} < y < 0$$

$$cotg\, z = x < 0 \overset{(2)}{\Longrightarrow} -\frac{\pi}{2} < z < 0$$

$tg\, y = cotg\, z \Longrightarrow \| y = z = \frac{\pi}{4} \|$. Logo,

$$f(x) = \text{arctg}\, x + \text{arctg}\, \frac{1}{x}$$

$$= y + z = -\frac{\pi}{4} + \left(-\frac{\pi}{4}\right) = -\frac{\pi}{2}$$

$$\boxed{f(x) = -\frac{\pi}{2}, \quad x \in (-\infty, 0)}$$

ii) Seja $x \in (0, +\infty)$.

Então

$$\text{tg}\, y = x > 0 \overset{①}{\Longrightarrow} 0 < y < \frac{\pi}{2}$$

$$\text{cotg}\, z = x > 0 \overset{②}{\Longrightarrow} 0 < z < \frac{\pi}{2}$$

e

$$\text{tg}\, y = \text{cotg}\, z \quad \text{com} \quad \begin{array}{l} 0 < y < \frac{\pi}{2} \\ 0 < z < \frac{\pi}{2} \end{array} \Longrightarrow \boxed{y = z = \frac{\pi}{4}}.$$

Logo,

$$f(x) = \text{arctg}\, x + \text{arctg}\, \frac{1}{x}$$

$$= y + z$$

$$= \frac{\pi}{4} + \frac{\pi}{4} = \frac{\pi}{2}$$

$\therefore$

$$\boxed{f(x) = \frac{\pi}{2}, \quad x \in (0, +\infty)}$$

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8. $\arcsin \frac{m-1}{m+1} = \arccos \frac{2\sqrt{m}}{m+1} \qquad (m \geq 0)$

Seja $y = \arcsin \frac{m-1}{m+1}$ $\therefore$ $y \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ (*)

$\therefore$ $\sin y = \frac{m-1}{m+1}$

Mas $\cos^2 y = 1 - \sin^2 y = 1 - \frac{(m-1)^2}{(m+1)^2}$

$$= \frac{(m+1)^2 - (m-1)^2}{(m+1)^2}$$

$$= \frac{m^2 + 2m + 1 - (m^2 - 2m + 1)}{(m+1)^2}$$

$$= \frac{\cancel{m^2} + 2m \cancel{+1} - \cancel{m^2} + 2m \cancel{-1}}{(m+1)^2}$$

$$= \frac{4m}{(m+1)^2}$$

$\therefore$

$\cos y = \pm \frac{2\sqrt{m}}{m+1}$

Mas $y \in [-\frac{\pi}{2}, \frac{\pi}{2}] \Rightarrow \cos y > 0$ $\therefore$ $\cos y = \frac{2\sqrt{m}}{m+1}$

$\therefore$ $y = \arccos \frac{2\sqrt{m}}{m+1}$

(**)

De (*) e (**):

$$\boxed{\arcsin \frac{m-1}{m+1} = \arccos \frac{2\sqrt{m}}{m+1}}$$

9.

$\arcsin x + \arcsin 2x = \frac{\pi}{2}$, $x \in (0,1)$

(*)

Seja

$\begin{cases} y = \arcsin x \rightarrow \sin y = x & (2*) \\ z = \arcsin 2x \rightarrow \sin z = 2x & (3*) \end{cases}$

De (*) : $y + z = \pi/2$ (4*)

Mas, para definirmos $z = \arcsin 2x$ devemos ter x tal que

$-1 \leq 2x \leq 1$

$\therefore -\frac{1}{2} \leq x \leq \frac{1}{2}$

Sendo $x \in (0,1)$ temos então que

$0 < x \leq \frac{1}{2}$ (5*)

De (2*) e (3*) : $\sin z = 2 \sin y$ (6*)

Mas de (*) :

$y + z = \frac{\pi}{2}$ $\therefore$ $z = \frac{\pi}{2} - y$ (7*)

De (7*) e (6*) :

$\sin\left(\frac{\pi}{2} - y\right) = 2 \sin y$

$\underbrace{\sin\frac{\pi}{2}}_{1} \cos y - \sin y \underbrace{\cos\frac{\pi}{2}}_{0} = 2 \sin y$ $\therefore$

9. cont

$\cos y = 2 \sin y$

$\therefore \quad \dfrac{\sin y}{\cos y} = \dfrac{1}{2}$

Mas $\left.\begin{array}{l} \sin y = x \in (0, \frac{1}{2}] \\ y = \arcsin x \end{array}\right\} \Rightarrow y \in (0, \frac{\pi}{6}]$

Daí $\dfrac{\sin y}{\cos y} = \dfrac{1}{2} \Rightarrow \dfrac{\sin y}{\sqrt{1-\sin^2 y}} = \dfrac{1}{2}$

$\therefore \quad \dfrac{\sin^2 y}{1-\sin^2 y} = \dfrac{1}{4}$

$\therefore \quad 4 \sin^2 y = 1 - \sin^2 y$

$5 \sin^2 y = 1$

$\therefore \quad \underbrace{\sin y}_{} = \dfrac{1}{\sqrt{5}} \quad (\text{pois } y \in (0, \frac{\pi}{6}))$

$// \ x = \dfrac{1}{\sqrt{5}} \ //$

10. $0 \le \arcsin \frac{a-1}{a+1} \le \frac{\pi}{2}$ (*)

Seja $x = \arcsin \frac{a-1}{a+1} \rightarrow \sin x = \frac{a-1}{a+1}$

$y = \operatorname{arctg} \frac{1}{2\sqrt{a}} \rightarrow \operatorname{tg} y = \frac{1}{2\sqrt{a}}$

Daí

$\operatorname{tg}\left(\arcsin \frac{a-1}{a+1} + \arctan \frac{1}{2\sqrt{a}}\right) =$

$= \operatorname{tg}(x + y)$

$= \dfrac{\operatorname{tg} x + \operatorname{tg} y}{1 - \operatorname{tg} x \operatorname{tg} y}$

De (*) : $0 \le x \le \frac{\pi}{2} \rightarrow \cos x = \sqrt{1 - \sin^2 x}$

Daí // $\operatorname{tg} x = \dfrac{\sin x}{\cos x} = \dfrac{\sin x}{\sqrt{1-\sin^2 x}} = \dfrac{\frac{a-1}{a+1}}{\sqrt{1-\left(\frac{a-1}{a+1}\right)^2}}$

$= \dfrac{\frac{a-1}{a+1}}{\sqrt{\frac{(a+1)^2-(a-1)^2}{(a+1)^2}}} = \dfrac{\frac{a-1}{a+1}}{\sqrt{\frac{\cancel{a^2}+2a+\cancel{1}-\cancel{a^2}+2a-\cancel{1}}{(a+1)^2}}}$

$= \dfrac{\frac{a-1}{a+1}}{\sqrt{\frac{4a}{(a+1)^2}}} = \dfrac{\frac{a-1}{\cancel{a+1}}}{\frac{2\sqrt{a}}{\cancel{a+1}}} = \dfrac{a-1}{2\sqrt{a}}$ //

$(a>0)$

10. cont.

Dar:

$$\operatorname{tg}\left(\arcsin\frac{a-1}{a+1} + \arctan\frac{1}{2\sqrt{a}}\right) =$$

$$= \frac{\operatorname{tg}x + \operatorname{tg}y}{1 - \operatorname{tg}x\operatorname{tg}y} = \frac{\frac{a-1}{2\sqrt{a}} + \frac{1}{2\sqrt{a}}}{1 - \frac{a-1}{2\sqrt{a}}\frac{1}{2\sqrt{a}}}$$

$$= \frac{\frac{a}{2\sqrt{a}}}{1 - \frac{a-1}{4a}}$$

$$= \frac{\frac{a}{2\sqrt{a}}}{\frac{4a-a+1}{4a}}$$

$$= \frac{\frac{a}{2\sqrt{a}}}{\frac{3a+1}{4a}} = \frac{a}{\cancel{2}\sqrt{a}} \cdot \frac{\cancel{4}a}{3a+1}$$

$$= \frac{2a^2}{(3a+1)}\frac{1}{\sqrt{a}} = \frac{2a^2\sqrt{a}}{(3a+1)a}$$

$$= \frac{2a\sqrt{a}}{3a+1}$$

11. $\operatorname{arctg} x + \arctan \dfrac{x}{x+1} = \dfrac{\pi}{4}$ $\quad (x \neq -1)$

Seja $\quad y = \operatorname{arctg} x \longrightarrow \operatorname{tg} y = x \quad (*)$

$\qquad y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$\qquad z = \operatorname{arctg} \dfrac{x}{x+1} \longrightarrow \operatorname{tg} z = \dfrac{x}{x+1} \quad (**)$

$\qquad z \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$$\therefore \quad y + z = \frac{\pi}{4}$$

$$\therefore \quad \operatorname{tg}(y+z) = \operatorname{tg}\frac{\pi}{4}$$

$$\frac{\operatorname{tg} y + \operatorname{tg} z}{1 - \operatorname{tg} y \operatorname{tg} z} = 1$$

$$\therefore \quad \frac{x + \frac{x}{x+1}}{1 - \frac{x^2}{x+1}} = 1$$

$$\therefore \quad \frac{\frac{x^2+x+x}{\cancel{x+1}}}{\frac{x+1-x^2}{\cancel{x+1}}} = 1$$

$$\therefore \quad \frac{x^2+2x}{-x^2+x+1} = 1$$

$$\therefore \quad x^2 + 2x = -x^2 + x + 1$$

$$2x^2 + x - 1 = 0$$

$$x = \frac{-1 \pm \sqrt{1+8}}{4} = \frac{-1 \pm 3}{4} = \begin{cases} -1 \\ \text{ou} \\ \frac{1}{2} \end{cases}$$

Mas $x \neq -1$

$\therefore$ $\boxed{x = \frac{1}{2}}$

12.

$$\sec\left(\operatorname{arctg}\frac{1}{1+e^x} - \operatorname{arctg}(1-e^x)\right) = \frac{\sqrt{5}}{2}$$

Seja

$$y = \operatorname{arctg}\frac{1}{1+e^x} \Longrightarrow \begin{cases} \operatorname{tg} y = \frac{1}{1+e^x} \\ y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \end{cases}$$

$$z = \operatorname{arctg}(1-e^x)$$

$$\Longrightarrow \begin{cases} \operatorname{tg} z = 1-e^x \\ z \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \end{cases}$$

1°.

$$\sec(y-z) = \frac{\sqrt{5}}{2} \quad (*)$$

2°.

$$\sec^2(y-z) = \frac{5}{4}$$

$$1 + \operatorname{tg}^2(y-z) = \frac{5}{4}$$

$$\left(\operatorname{tg}(y-z)\right)^2 = \frac{1}{4}$$

$$\left(\frac{\operatorname{tg} y - \operatorname{tg} z}{1 + \operatorname{tg} y \operatorname{tg} z}\right)^2 = \frac{1}{4}$$

$$\left(\frac{\frac{1}{1+e^x} - (1-e^x)}{1 + \frac{1}{1+e^x}(1-e^x)}\right)^2 = \frac{1}{4}$$

$$\left(\frac{\dfrac{1-(1-e^x)(1+e^x)}{\cancel{1+e^x}}}{\dfrac{1+\cancel{e^x}+1-\cancel{e^x}}{\cancel{1+e^x}}}\right)^2 = \frac{1}{4}$$

$$\left[\frac{1-(1-e^{2x})}{2}\right]^2 = \frac{1}{4}$$

$$\left(\frac{e^{2x}}{2}\right)^2 = \frac{1}{4}$$

$$\frac{e^{4x}}{4} = \frac{1}{4} \Rightarrow e^{4x} = 1 \Rightarrow \underline{\underline{x=0}}$$

Conferindo:

$\operatorname{tg} z = 1 - e^x \xrightarrow{x=0} \operatorname{tg} z = 0$

$z \in (-\frac{\pi}{2}, \frac{\pi}{2})$

$\therefore \underline{\underline{z=0}}$ pois $z \in (-\frac{\pi}{2}, \frac{\pi}{2})$

$\operatorname{tg} y = \frac{1}{1+e^x} \xrightarrow{x=0} \operatorname{tg} y = \frac{1}{2}$

$y \in (-\frac{\pi}{2}, \frac{\pi}{2})$

$\sec y = \sqrt{1+\operatorname{tg}^2 y}$

pois $\sec y \geq 0$ e $y \in (-\frac{\pi}{2}, \frac{\pi}{2})$

$\therefore \sec y = \underline{\underline{\frac{\sqrt{5}}{2}}}$

Daí, em (*) (a eq. antes de elevarmos ao quadrado):

$\sec(y-3) = \sec(y-0) = \sec y = \frac{\sqrt{5}}{2}$

satisfaz (*).

Assim $\underline{x=0}$ é solução.

— // — // —

13. $a \in (-\frac{\pi}{2}, \frac{\pi}{2})$

(*) $\arctan(\sqrt{2}-1+\frac{e^t}{2}) + \arctan(\sqrt{2}-1-\frac{e^t}{2}) = a$

Seja $y = \operatorname{arctg}(\sqrt{2}-1+\frac{e^t}{2})$

$\therefore \operatorname{tg} y = \sqrt{2}-1+\frac{e^t}{2}, \quad y \in (-\frac{\pi}{2}, \frac{\pi}{2})$

$z = \operatorname{arctg}(\sqrt{2}-1-\frac{e^t}{2})$

$\therefore \operatorname{tg} z = \sqrt{2}-1-\frac{e^t}{2}, \quad z \in (-\frac{\pi}{2}, \frac{\pi}{2})$

(*) resulta em:

$y + z = a$

13. cont.

$\therefore \quad tg(y+z) = tg\,a$

$$\frac{tg\,y + tg\,z}{1 - tg\,y\; tg\,z} = tg\,a$$

$$\frac{\sqrt{2}-1+\cancel{\frac{e^t}{2}}+\sqrt{2}-1-\cancel{\frac{e^t}{2}}}{1-\left(\sqrt{2}-1+\frac{e^t}{2}\right)\left(\sqrt{2}-1-\frac{e^t}{2}\right)} = tg\,a$$

$$\frac{2(\sqrt{2}-1)}{1-\left[(\sqrt{2}-1)^2-\frac{e^{2t}}{4}\right]} = tg\,a$$

$$\frac{2(\sqrt{2}-1)}{1-\left[2-2\sqrt{2}+1-\frac{e^{2t}}{4}\right]} = tg\,a$$

$$\frac{2(\sqrt{2}-1)}{-2+2\sqrt{2}+\frac{e^{2t}}{4}} = tg\,a$$

$$\frac{2(\sqrt{2}-1)}{tg\,a} = -2+2\sqrt{2}+\frac{e^{2t}}{4}$$

$$\frac{2(\sqrt{2}-1)}{tg\,a} - 2(\sqrt{2}-1) = \frac{e^{2t}}{4}$$

$$2(\sqrt{2}-1)\left(\frac{1}{tg\,a}-1\right) = \frac{e^{2t}}{4}$$

$\longrightarrow$

$$8(\sqrt{2}-1)\left(\frac{1}{tg\,a}-1\right)=e^{2x}$$

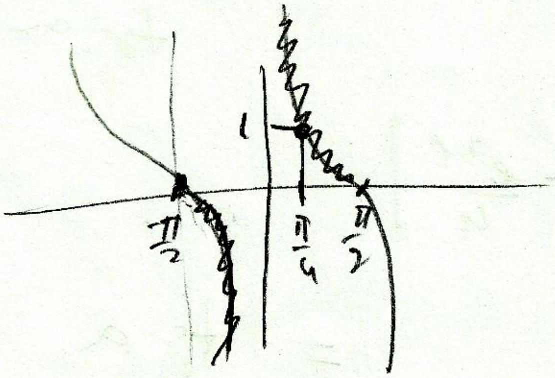
$$\ln 8(\sqrt{2}-1)\left(\frac{1}{tg\,a}-1\right)=2x$$

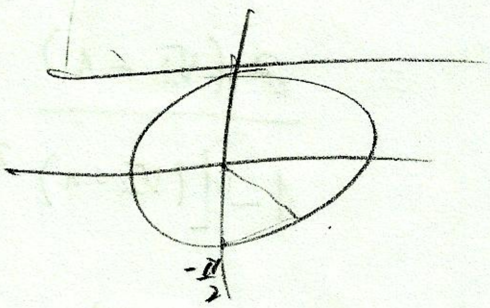
$$x=\frac{1}{2}\ln \underbrace{8(\sqrt{2}-1)}_{>0}\left(\frac{1}{tg\,a}-1\right), \quad a\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$$

Para tener x definido debemos ter

$$\frac{1}{tg\,a}-1>0$$

$$\cot g\,a>1, \quad a\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$$





$$\left.\begin{array}{l}\cot g\,a=1\\ -\frac{\pi}{2}<a<\frac{\pi}{2}\end{array}\right\} \Rightarrow a\in\frac{\pi}{4}$$

Daí $\cot g\,a>1 \Rightarrow \underline{a\in\left(0,\frac{\pi}{4}\right)}$

14.

$$\begin{cases} x = \arcsin \frac{b}{a} , & |a| > |b| \\ & 0 \leq x \leq \frac{\pi}{2} . \end{cases}$$

$$\left[x = \frac{1}{2} \arcsin \frac{2b\sqrt{a^2-b^2}}{a^2} \right]$$

Seja $x = \arcsin \frac{b}{a}$

$\therefore \sin x = \frac{b}{a}$

Mes

$$\begin{aligned} \sin 2x &= 2 \sin x \cos x \\ &= 2 \underbrace{\sin x}_{} \sqrt{1 - \sin^2 x} \quad \text{pois } 0 \leq x \leq \tfrac{\pi}{2} \\ &= 2 \frac{b}{a} \sqrt{1 - \frac{b^2}{a^2}} \\ &= 2 \frac{b}{a} \sqrt{\frac{a^2-b^2}{a^2}} \\ &= \frac{2b}{a} \frac{\sqrt{a^2-b^2}}{|a|} \end{aligned}$$

$$2x = \arcsin \frac{2b\sqrt{a^2-b^2}}{a|a|}$$

$$\Big/\!\Big/ \; x = \frac{1}{2} \arcsin \frac{2b\sqrt{a^2-b^2}}{a|a|} \; \Big/\!\Big/$$

15.

$$\operatorname{arctg} \frac{1+x}{2} + \operatorname{arctg} \frac{1-x}{2} \geq \frac{\pi}{4}$$

Seja $y = \operatorname{arctg} \frac{1+x}{2}$

$$\begin{cases} \operatorname{tg} y = \frac{1+x}{2} \\ y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \end{cases}$$

$z = \operatorname{arctg} \frac{1-x}{2}$

$$\begin{cases} \operatorname{tg} z = \frac{1-x}{2} \\ z \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \end{cases}$$

Assim:

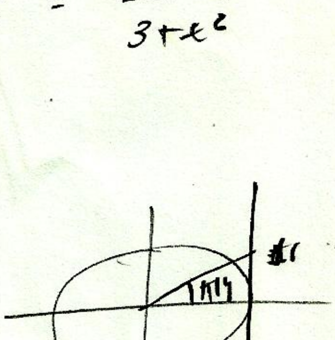
$$\operatorname{arctg} \frac{1+x}{2} + \operatorname{arctg} \frac{1-x}{2} = y + z.$$

Mas

$$\operatorname{tg}(y+z) = \frac{\operatorname{tg} y + \operatorname{tg} z}{1 - \operatorname{tg} y \operatorname{tg} z} = \frac{\frac{1+x}{2} + \frac{1-x}{2}}{1 - \frac{1+x}{2} \cdot \frac{1-x}{2}}$$

$$= \frac{1}{1 - \frac{1-x^2}{4}} = \frac{1}{\frac{4-1+x^2}{4}} = \frac{4}{3+x^2}$$

$$\therefore \operatorname{tg}(y+z) = \frac{4}{3+x^2} > 0$$



Mas $y+z \geq \frac{\pi}{4}$

$$\Longrightarrow \operatorname{tg}(y+z) = \frac{4}{3+x^2} \geq 1$$

15. cont.

$4 \geq 3 + x^2$

$0 \geq -1 + x^2 \implies \| x \in [-1, 1] \|$

16. $\sin\left(2 \operatorname{arc}\cot \frac{4}{3}\right) + \cos\left(2 \operatorname{arc}\operatorname{cosec} \frac{5}{4}\right) = \frac{17}{25}$

Sejam
$$\begin{cases} y = \operatorname{arc}\cot \frac{4}{3} \longrightarrow \begin{cases} \cot y = \frac{4}{3} \\ y \in (0, \pi) \end{cases} \\ z = \operatorname{arc}\operatorname{cosec} \frac{5}{4} \longrightarrow \begin{cases} \operatorname{cosec} z = \frac{5}{4} \\ z \in (-\pi, -\frac{\pi}{2}] \cup (0, \frac{\pi}{2}] \end{cases} \end{cases}$$

$\sin\left(2 \operatorname{arc}\cot \frac{4}{3}\right) + \cos\left(2 \operatorname{arc}\operatorname{cosec} \frac{5}{4}\right) =$

$= \sin 2y + \cos 2z$

$= 2 \sin y \cos y + \cos^2 z - \sin^2 z$

Mas

$\cot y = \frac{4}{3}$, $y \in (0, \pi)$ $\Longrightarrow y \in (0, \frac{\pi}{2})$

$\frac{4}{3} = \cot y = \frac{\sqrt{1 - \sin^2 y}}{\sin y}$

$\frac{16}{9} = \frac{1 - \sin^2 y}{\sin^2 y} \Longrightarrow 16 \sin^2 y = 9 - 9 \sin^2 y$

$25 \sin^2 y = 9$

$\sin^2 y = \frac{9}{25}$

$\| \sin y = \frac{3}{5} \|$ pois $y \in (0, \frac{\pi}{2})$

$\therefore \| \cos y = \sqrt{1 - \frac{9}{25}} = \frac{4}{5} \|$

$\operatorname{cosec} z = \frac{5}{4} \Rightarrow // \sin z = \frac{4}{5} // > 0 \Rightarrow$

$z \in (-\pi, -\frac{\pi}{2}] \cup (0, \frac{\pi}{2}]$

$z \in [0, \frac{\pi}{2}]$

$$\cos z = \sqrt{1 - \sin^2 z}$$

$$= \sqrt{1 - \frac{16}{25}} = \frac{3}{5}$$

$\therefore // \cos z = \frac{3}{5} //$

Dar atunci:

$$\sin\left(2 \operatorname{arc\,ctg} \tfrac{4}{3}\right) + \cos\left(2 \operatorname{arc\,cosec} \tfrac{5}{4}\right) =$$

$$= 2 \sin y \cos y + \cos^2 z - \sin^2 z$$

$$= 2 \cdot \frac{3}{5} \cdot \frac{4}{5} + \frac{9}{25} - \frac{16}{25}$$

$$= \frac{24}{25} + \frac{9}{25} - \frac{16}{25} = \frac{24 - 7}{25} = \frac{17}{25} //$$