Unifying the Galilei and the Special Relativity: The Galilei electrodynamics

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Abstract

We obtain the Galilei electrodynamics developed by M. Levy-Leblond and M. Le Bellac as an application of a recent scheme we have proposed for unifying the Galilei and the special relativity, which is based on a reinterpretation of the absolute time of the Galilei relativity. We achieve this by defining two coordinate systems for spacetime that we have called the Galilei and the Lorentz systems, and we show how the relation between those systems allows us to develop a tensor calculus that transfer the Maxwell equations of the classical electrodynamics to the Galilei system. Then, by using a suitable low velocity limit, we show how the Maxwell equations in the Galilei system become the fundamental equations of the Galilei electrodynamics.

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1 Introduction

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$\lrcorner$ $x^\alpha \equiv (x^0, x^i) := (ct, \vec{x})$: coordenadas (naturalmente contravariante)

$\lrcorner$ $d\tau^2 := dt^2 - \frac{1}{c^2} d\vec{x}^{\,2}$: tempo-próprio

$$= dt^2 \left(1 - \frac{1}{c^2}\left(\frac{d\vec{x}}{dt}\right)^2\right)$$

$$\therefore\ d\tau = \sqrt{1 - \frac{1}{c^2}\left(\frac{d\vec{x}}{dt}\right)^2}\, dt$$

$\lrcorner$ $u^\alpha := \frac{dx^\alpha}{d\tau} := \left(\frac{dx^0}{d\tau}, \frac{dx^i}{d\tau}\right)$: 4-velocidade (contravariante)

$\lrcorner\lrcorner$ $\frac{d\vec{x}}{d\tau} = \frac{d\vec{x}}{dt}\frac{dt}{d\tau} = \frac{d\vec{x}}{dt}\frac{1}{\sqrt{1-\frac{1}{c^2}\left(\frac{d\vec{x}}{dt}\right)^2}} \equiv \gamma_v \vec{v}$

$\lrcorner\lrcorner$ $\frac{dx^0}{d\tau} = \frac{cdt}{d\tau} = c\frac{1}{\sqrt{1-\frac{1}{c^2}\left(\frac{d\vec{x}}{dt}\right)^2}} \equiv \gamma_v c$

onde

$\gamma_v := \sqrt{1 - \frac{v^2}{c^2}}$

$\vec{v} := \frac{d\vec{x}}{dt}$

$$\therefore\ \boxed{u^\alpha = (\gamma_v c, \gamma_v \vec{v})}$$

$\lrcorner$ $\eta_{\alpha\beta} : \eta_{00} = -1,\ \eta_{0i} = \eta_{i0} = 0,\ \eta_{ij} = \delta_{ij}$: métrica

$\lrcorner$ $x_\alpha := \eta_{\alpha\beta} x^\beta$

$x_\alpha \equiv (x_0, x_i) := (\eta_{0\beta}x^\beta, \eta_{i\beta}x^\beta) = (-x^0, x^i) = (-ct, \vec{x})$

$\lrcorner$ $u_\alpha = \eta_{\alpha\beta}u^\beta = \eta_{\alpha\beta}\frac{dx^\beta}{d\tau} = \frac{d(\eta_{\alpha\beta}x^\beta)}{d\tau} = \frac{dx_\alpha}{d\tau}$

$u_\alpha \equiv (u_0, u_i) = (\eta_{0\beta}u^\beta, \eta_{i\beta}u^\beta) = (-u^0, u^i) = (-\gamma_v c, \gamma_v \vec{v})$

$$\therefore\ \boxed{u_\alpha = (-\gamma_v c, \gamma_v \vec{v})}$$

$\lrcorner$ $\vec{E} = (E_i),\ \vec{B} = (B_i)$

Definimos um tensor antissimétrico:

$F^{\mu\nu} : F^{0i} := E_i,\ F^{ij} := \epsilon_{ijk} B_k$

$\therefore\ E_1 = F^{01},\ E_2 = F^{02},\ E_3 = F^{03}$

$B_1 = F^{23},\ B_2 = F^{31},\ B_3 = F^{12}$

The rising of Einstein's special relativity in 1905 has placed since then the Galilei relativity as an approximation of the former theory when we consider inertial reference frames having relative velocity with a magnitude that is sufficiently low as compared to the speed of light. Among the roots of special relativity we find classical electrodynamics, whose main equations are invariant under the Lorentz group that includes the Lorentz boost as a particular case (here, we refer the Lorentz boost simply by Lorentz transformation). Here, in the same way as the kinematical equations of special relativity in a certain low velocity limit become the kinematical equations of the Galilei relativity, one could ask what theory emerges as the low velocity limit of the Maxwell equations of classical electrodynamics? Would it exhibit invariance under a sort of Galilei transformation for the fields, the charge and the current densities? Those questions were addressed long ago by M. Levy-Leblond and M. Le Bellac [1] who obtained the so-called Galilei electrodynamics as the correct Galilean limit of the classical electrodynamics, which constitutes a consistent non-relativistic limit for the Maxwell's equations. In their work they showed this non-relativistic limit is not concerned with simply taking $c \to \infty$ in the Maxwell's equations since the explicit presence of the speed of light c in these equations depends on the choice of the system of units being used. In fact, in their attempt to solve those subtleties, Levy Leblond and Le Bellac argued on the convenience of the SI system in order to set a correct non-relativistic limit for the Maxwell's equations together with some restrictions on the electric and magnetic fields that encompasses two distinct models, the so-called *electric* and *magnetic* limits (in fact, there is also a third model, the *General Galilean Electromagnetism*).

One of the interests on studying the Galilean limit of the Maxwell's equations is to provide a criteria to understand which electromagnetic effects can be reasonably described by a nonrelativistic theory and then to exhibit such a theory, and also to distinguish those phenomena having their description only in a relativistic context [1]. Another interest is to provide a suitable Galilei transformation for the electromagnetic fields that corrects some low velocity formulas given in some textbooks. Besides that, as a natural development of these ideas, some authors have continued the study of the Galilei electrodynamics considering other aspects, for instance, some applications in quantum mechanics and superconductivity [2]; the form of the electromagnetic potentials and the gauge conditions in the Galilean limits [2], [3], [4], and so on. Recently, some developments [5], [6] brought new insight into the original work of [1], where the authors considered *alternative* ways to obtain the electric and the magnetic limits. Our present work falls into this category as we intend to show how the electric, the magnetic, and the general Galilei electrodynamics obtained in [1] follow as a natural consequence of a recent scheme we proposed to unify the Galilei and the special relativity as we now describe.

In a previous work [7], we presented a method for unifying the Galilei and the special relativity into a single model. This unification was performed through the introduction of an absolute time that plays the role of the time variable of the Galilei relativity, together with the local time t, which is the ordinary time of the special relativity. In terms of these time variables there are two views one can employ to describe events, each one being adapted to the particularities of either the Galilei or the special relativity. Thus, events described within the realm of the Galilei relativity are defined by a coordinate set $\{\tau, \vec{x}\}$, while special relativity considers for set $\{t, \vec{x}\}$ (we assume the space coordinates to be the same in both views). As we have shown in [7], in order to combine the Galilei and the special relativity into one single model we first need to extend the previous variables set to $\{\tau, t, \vec{x}\}$. Then, given two inertial frames S, S' moving with relative velocity $\vec{v}$ we assume between the respective sets $\{\tau, t, \vec{x}\}$, $\{\tau', t', \vec{x}\,'\}$ the relations

$$\begin{aligned} &\tau' = \tau \\ &\vec{x}\,' = \vec{x} - \vec{v}\tau \\ &c^2 t'^2 - \vec{x}\,'^2 = c^2 t^2 - \vec{x}^2 \ . \end{aligned} \tag{1}$$

As a result, assuming a linear relation between t and t' we obtain

$$\tau = (1-a)\frac{\vec{x}\cdot\vec{v}}{v^2} + \sqrt{a^2-1}\,\frac{c}{v}\,t = (1-a)\frac{\vec{x}\,'\cdot(-\vec{v})}{v^2} + \sqrt{a^2-1}\,\frac{c}{v}\,t' = \tau' \tag{2}$$

which provides a relation between the absolute time τ and the local time t, t', with a being an arbitrary parameter.

Here, in our current work, we will show how relation (2) allows us to introduce two sets of coordinates systems, denoted by X^μ_G, X^μ_L, which define the *Galilei* and the *Lorentz* systems, each one encoding respectively the transformation properties that are common either to the Galilei or to the special relativity. Furthermore, we assume the Maxwell's equations as naturally described with respect to the Lorentz system. Then, using $\partial X^\mu_G/\partial X^\nu_L$ as transformations coefficients we transfer all fields and the Maxwell's equations to the Galilei system. In this way we introduce from the electric and the magnetic fields of the standard Maxwell theory, e.g. $\vec{E}_L$, $\vec{B}_L$ (thought as components of a tensor $\mathcal{F}_{L\mu\nu}$ or $\mathcal{F}^{\mu\nu}_L$) the corresponding Galilean analogues, $\vec{E}_G, \vec{B}_G$, and, in an similar way, we set the Galilean transformations of $\vec{E}_G$, $\vec{B}_G$ from the Lorentz transformation of $\vec{E}_L, \vec{B}_L$. Therefore, the equations satisfied by the Galilean fields are obtained directly from the Maxwell equations by replacing $\vec{E}_L, \vec{B}_L$ by their expressions in terms of $\vec{E}_G, \vec{B}_G$ together with the transformation expressing the derivatives relative to X^μ_L in terms of the derivatives relative to X^μ_G. Once this is performed, we are ready to show how the electric and the magnetic limits employed in [1] arise from the corresponding Galilean form of the Maxwell equations when we take a suitable limit case. This indicates that the unification of the Galilei and the special relativity exhibited in our previous work [7] and based on the fundamental relation between τ, t expressed by (2) extends beyond the kinematical aspects of both relativities, reproducing the correct Galilean limit of the relativistic Maxwell electrodynamics as discovered by Levy Leblond and Le Bellac.

Our work is organized as follows. In section **2** we set our notations and review the basics aspects of the Maxwell electrodynamics that we will use in the subsequent sections. In section **3** we review the main elements of [7]. We base our analysis on a class of transformations parameterized by a real parameter a, with $|a| > 1$, that we call *Generalized Lorentz Transformations* (GLT) and that follows from the conditions given in (1). Then, we formulate Maxwell electrodynamics as being invariant under the GLT. Compared to the standard Maxwell theory, this brings a modification to the form of the transformations of the electromagnetic fields but doesn't change the form of the Maxwell equations. We also show that the GLT includes the ordinary Lorentz transformation as a particular case, and in such case the GLT transformation for the fields and the four-current become the usual Lorentz transformation for the fields and the four-current of the standard Maxwell theory. In section **5** we explain how to perform the Galilean limit of our model. We also employ the same approximation used by Leblond and Le Bellac [1], for instance, $c|\rho| \ll |\vec{j}|$ and $|\vec{E}| \ll c|\vec{B}|$ in the magnetic limit, and $c|\rho| \gg |\vec{j}|$ and $|\vec{E}| \gg c|\vec{B}|$ in the electric limit. Besides that, our limit is obtained employing the limit $\frac{1}{c^2} \to 0$ after making a Taylor expansion in terms of $\frac{\tilde{v}}{c}$ of the parameter a. In section **7** we develop the Galilei electrodynamics employing the "tensor calculus" determined from the relation between the two coordinate systems X_G^μ and X_L^μ. Then, we show how the three Galilean models of [1] arise by applying the Galilean limit of section **5**. In particular, we pay a special attention to the third model of [1], the general Galilean electromagnetism, that is formulated in terms of the four fields $\vec{E}_L, \vec{B}_L, \vec{D}_L, \vec{H}_L$. After defining the corresponding Galilei fields $\vec{E}_G, \vec{B}_G, \vec{D}_G, \vec{H}_G$ we show that it is possible to derive appropriate constitutive relations among the Galilean fields that preserve the Galilei invariance, a feature that is not possible in the treatment of Leblond and Le Bellac.

In our work we will use only the CGS system of units. The need for that is because we take the electric and the magnetic fields as components of tensors $\mathcal{F}_{\mu\nu}$, $\mathcal{F}^{\mu\nu}$, which is suitably introduced within the CGS system.

2 Maxwell electrodynamics

In order to fix our notation, we will recall briefly some aspects of the standard Maxwell electrodynamics [8]. Spacetime is described by coordinates $x^\mu \equiv (x^0, x^i) := (ct, \vec{x})$, with c being the speed of ligth in vacuum. We also write $x_\mu := \eta_{\mu\nu} x^\nu = (ct, -\vec{x})$. The electric and the magnetic fields are accommodated as components of two antisymmetric tensors $\mathcal{F}^{\mu\nu}$, $\mathcal{F}_{\mu\nu}$ according to

$$
\begin{aligned}
B_i &= -\tfrac{1}{2}\,\epsilon_{ijk}\mathcal{F}_{jk} = -\tfrac{1}{2}\,\epsilon_{ijk}\mathcal{F}^{jk} \\
E_i &= \mathcal{F}_{0i} = -\mathcal{F}^{0i}\ .
\end{aligned}
\tag{3}
$$

Then, the Maxwell equations in vacuum become

$$
\begin{array}{l}
\partial_\mu\mathcal{F}_{\nu\lambda}+\partial_\nu\mathcal{F}_{\lambda\mu}+\partial_\lambda\mathcal{F}_{\mu\nu}=0 \\
\partial_\mu\mathcal{F}^{\mu\nu}=\frac{4\pi}{c}J^\nu \\
J^\mu=(c\rho,\vec{j})
\end{array}
\quad\rightleftharpoons\quad
\begin{cases}
\vec{\nabla}\times\vec{E}+\frac{1}{c}\partial_t\vec{B}=0 \\
\vec{\nabla}\cdot\vec{B}=0 \\
\vec{\nabla}\cdot\vec{E}=4\pi\rho \\
\vec{\nabla}\times\vec{B}-\frac{1}{c}\partial_t\vec{E}=\frac{4\pi}{c}\vec{j}\ .
\end{cases}
\tag{4}
$$

In the presence of a material medium the previous Maxwell equations in vacuum must be changed due to extra contributions to the density of charge and current produced by the medium. Now, in addition to the electric and magnetic fields $\vec{E}$, $\vec{B}$, we also have the fields $\vec{D}$, $\vec{H}$ that are accommodated as components of another antisymmetric tensor $\mathcal{H}^{\mu\nu}$ according to

$$
\begin{aligned}
H_i &= -\tfrac{1}{2}\,\epsilon_{ijk}\mathcal{H}_{jk} = -\tfrac{1}{2}\,\epsilon_{ijk}\mathcal{H}^{jk} \\
D_i &= \mathcal{H}_{0i} = -\mathcal{H}^{0i}\ .
\end{aligned}
\tag{5}
$$

Here, Maxwell's equation in the presence of a medium becomes [8]

$$
\begin{array}{l}
\partial_\mu\mathcal{F}_{\nu\lambda}+\partial_\nu\mathcal{F}_{\lambda\mu}+\partial_\lambda\mathcal{F}_{\mu\nu}=0 \\
\partial_\mu\mathcal{H}^{\mu\nu}=\frac{4\pi}{c}J^\nu \\
J^\mu=(c\rho,\vec{j})
\end{array}
\quad\rightleftharpoons\quad
\begin{cases}
\vec{\nabla}\times\vec{E}+\frac{1}{c}\partial_t\vec{B}=0 \\
\vec{\nabla}\cdot\vec{B}=0 \\
\vec{\nabla}\cdot\vec{D}=4\pi\rho \\
\vec{\nabla}\times\vec{H}-\frac{1}{c}\partial_t\vec{D}=\frac{4\pi}{c}\vec{j}\ .
\end{cases}
\tag{6}
$$

In most cases the fields $\vec{D}$, $\vec{H}$ relate to the fields $\vec{E}$, $\vec{B}$ through the polarization and magnetization vectors, $\vec{P}$, $\vec{M}$ by the constitutive relations

$$
\begin{aligned}
\vec{D} &:= \vec{E}+4\pi\vec{P} \\
\vec{H} &:= \vec{B}-4\pi\vec{M}\ .
\end{aligned}
\tag{7}
$$

3 An overview of some previous results concerning the Galilei relativity and the special relativity

Here, we briefly recall some of the concepts introduced in [7], which we refer the reader for details. Let S and S' be two inertial reference frames moving with relative velocity

$\vec{v}$. Let P be an event. According to the Galilei relativity let us assume both observers record this event as $(\tau, \vec{x})$ and $(\tau, \vec{x}\,')$. The relation between their readings is

$$\vec{x}\,' = \vec{x} - \vec{v}\tau \ . \tag{8}$$

In order to relate the Galilei relativity with the framework of special relativity we assume the existence of another time variable, the local time of the special relativity, denoted by t. Now, let us assume that in terms of these coordinates each observer has recorded the event P as $(t, \vec{x})$ and $(t', \vec{x}\,')$. Here, the fundamental relation one imposes between these variables is that

$$c^2t^2 - \vec{x}^2 = c^2t'^2 - \vec{x}\,'^{\,2} \ . \tag{9}$$

Now, if we assume a linear relation between t and t' as

$$t' = at + b\vec{v} \cdot \vec{x} \tag{10}$$

with a and b arbitrary real coefficients, the fulfillment of equations (8, 9) by the set $\{\tau, t, t', \vec{x}, \vec{x}\,'\}$ and the assumptions stated in [7] gives

$$b = \sqrt{a^2 - 1}\frac{1}{vc} \tag{11}$$

and

$$\tau = (1-a)\frac{\vec{x} \cdot \vec{v}}{v^2} + \sqrt{a^2-1}\,\frac{c}{v}\,t = (1-a)\frac{\vec{x}\,' \cdot (-\vec{v})}{v^2} + \sqrt{a^2-1}\,\frac{c}{v}\,t' = \tau' \tag{12}$$

together with the so-called Generalized Lorentz Transformation (GLT)

$$\begin{cases} \vec{x}\,' = \vec{x} - (1-a)\frac{1}{v^2}\,\vec{x} \cdot \vec{v}\,\vec{v} - \sqrt{a^2-1}\frac{1}{v}\,c\,t\,\vec{v} \\ t' = at - \sqrt{a^2-1}\frac{1}{vc}\,\vec{x} \cdot \vec{v} \end{cases} \tag{13}$$

which represents a family of transformations parameterized by a real parameter a that is assumed to depend arbitrarily on the speed v between the frames, and to satisfy $a > 1$.

Since we are considering the absolute time τ and the physical time t we must distinguish between two velocities

$$\vec{v} = \frac{d\vec{x}_{SS'}}{d\tau}, \quad \vec{\tilde{v}} = \frac{d\vec{x}_{SS'}}{dt}$$

that are related by

$$\vec{\tilde{v}} = \vec{v}\frac{\sqrt{a^2-1}}{a}\frac{c}{v} \tag{14}$$

with $\vec{x}_{SS'}$ denoting the position of the origin of the frame S' as seen by frame S. In terms of $\widetilde{v}$ given in (14) the parameter a becomes

$$a = \frac{1}{\sqrt{1 - \frac{\widetilde{v}^2}{c^2}}} \equiv \gamma_{\widetilde{v}} \tag{15}$$

and the GLT transformation given in (13) assumes the form

$$\begin{cases} \vec{x}\,' = \vec{x} - (1 - \gamma_{\widetilde{v}}) \frac{\vec{x} \cdot \vec{\widetilde{v}}}{\widetilde{v}^2} \vec{\widetilde{v}} - \gamma_{\widetilde{v}} t \vec{\widetilde{v}} \\ t' = \gamma_{\widetilde{v}} \Big(t - \frac{\vec{x} \cdot \vec{\widetilde{v}}}{c^2} \Big) \end{cases} \tag{16}$$

which is the usual Lorentz transformation.

One should notice that the GLT given in (13) is parameterized in terms of the relative velocity $\vec{v} = \frac{d\vec{x}_{SS'}}{d\tau}$, while the ordinary Lorentz transformation is parameterized in terms of $\vec{\widetilde{v}} = \frac{d\vec{x}_{SS'}}{dt}$. The parameter a in the GLT is assumed to depend on the speed v, but as we see from (14), whatever might be the dependence of a with v we will always obtain the same expression for a in terms of $\widetilde{v}$ as given in (15). Therefore, we understand the standard Lorentz transformation as being *universal* in the sense that whenever we write a in terms of $\widetilde{v}$ the GLT becomes the ordinary Lorentz transformation.

Under the transformation (13) the electromagnetic fields and the four-current transform as

$$\begin{cases} \vec{E}' = a\vec{E} + (1-a)\frac{1}{v^2}\, \vec{v} \cdot \vec{E}\, \vec{v} + \sqrt{a^2 - 1}\frac{1}{v}\, \vec{v} \times \vec{B} \\ \vec{B}' = a\vec{B} + (1-a)\frac{1}{v^2}\, \vec{v} \cdot \vec{B}\, \vec{v} - \sqrt{a^2 - 1}\frac{1}{v}\, \vec{v} \times \vec{E} \end{cases} \tag{17}$$

$$\begin{cases} \vec{D}' = a\vec{D} + (1-a)\frac{1}{v^2}\, \vec{v} \cdot \vec{D}\, \vec{v} + \sqrt{a^2 - 1}\frac{1}{v}\, \vec{v} \times \vec{H} \\ \vec{H}' = a\vec{H} + (1-a)\frac{1}{v^2}\, \vec{v} \cdot \vec{H}\, \vec{v} - \sqrt{a^2 - 1}\frac{1}{v}\, \vec{v} \times \vec{D}\ . \end{cases} \tag{18}$$

$$\begin{cases} \rho' = a\rho - \sqrt{a^2 - 1}\frac{1}{cv} \vec{v} \cdot \vec{j} \\ \vec{j}\,' = \vec{j} + \Big(-\sqrt{a^2 - 1}\frac{c}{v}\rho - (1-a)\frac{1}{v^2}\, \vec{v} \cdot \vec{j} \Big) \vec{v}\ . \end{cases} \tag{19}$$

It is straightforward to check that the Maxwell equations (4), (6) are invariant under the transformations of the coordinates (13), the fields (17) and (18), and the four-current (19).

In terms of the velocity $\vec{\widetilde{v}}$ given in (14) the previous transformations become the usual transformations for the fields and 4-current under a Lorentz transformation

$$\vec{E}' = \gamma\vec{E} + (1-\gamma)\frac{1}{\tilde{\vec{v}}^2}\,\tilde{\vec{v}}\cdot\vec{E}\,\tilde{\vec{v}} + \frac{1}{c}\gamma\,\tilde{\vec{v}}\times\vec{B} \tag{20}$$

$$\vec{B}' = \gamma\vec{B} + (1-\gamma)\frac{1}{\tilde{\vec{v}}^2}\,\tilde{\vec{v}}\cdot\vec{B}\,\tilde{\vec{v}} - \frac{1}{c}\gamma\tilde{\vec{v}}\times\vec{E} \tag{21}$$

$$\vec{D}' = \gamma\vec{D} + (1-\gamma)\frac{1}{\tilde{\vec{v}}^2}\,\tilde{\vec{v}}\cdot\vec{D}\,\tilde{\vec{v}} + \frac{1}{c}\gamma\,\tilde{\vec{v}}\times\vec{H} \tag{22}$$

$$\vec{H}' = \gamma\vec{H} + (1-\gamma)\frac{1}{\tilde{\vec{v}}^2}\,\tilde{\vec{v}}\cdot\vec{H}\,\tilde{\vec{v}} - \frac{1}{c}\gamma\tilde{\vec{v}}\times\vec{D} \tag{23}$$

$$\rho' = \gamma\Big(\rho - \frac{1}{c^2}\tilde{\vec{v}}\cdot\vec{j}\Big) \tag{24}$$

$$\vec{j}\,' = \vec{j} - (1-\gamma)\frac{1}{\tilde{\vec{v}}^2}\tilde{\vec{v}}\cdot\vec{j}\,\tilde{\vec{v}} - \gamma\rho\tilde{\vec{v}}\,. \tag{25}$$

Remark: Equation (12) may be seen as an operational definition for the absolute time τ that is determined by the measurement of the local time t, t', and the space coordinates where the event occurred $\vec{x}, \vec{x}\,'$. Conversely, (12) may also be seen as an expression for the local time in terms of the absolute time and the relative velocity of the frames, an interpretation that suggests the local time depends on the state of motion of the observers (in order to explicitly indicate this we could have written t and t' as $t_{SS'}$, $t'_{SS'}$). As a consequence of this interpretation the local time doesn't attain the meaning of an intrinsic quantity defined uniquely with respect to a single frame. This view is also indicated from the idealized form on how the local time is established. In fact, the instants t and t' are recorded by clocks that are placed in the positions where the event occurred. Since these clocks (at rest relative to the same frame) are all synchronized and arranged in such way as to mark $t = t' = 0$ when the origins of the reference systems coincide, it is plausible to admit their functioning is somehow adjusted to the peculiarities of the *relative* motion between the frames. A similar and more complete discussion on this issue is given by Horwitz, Arshansky and Elitzur in [11] where they distinguished between two aspects of time, one that considers time as an extra dimension in a four dimensional space, and the other that envisages time as associated to a change or development, which they call *process time*, having the particularity that *"in relativity then, the time at which an event occurs depends on the state of motion of the frame (and the clocks attached to it)"* (pg. 1163). They go a step further in their discussion and analyze the role of a generating apparatus that records cycles that allow for the counting of time and observe that *"although it is a generally accepted notion that an event is described by its spacetime coordinates x and t alone, we see that the state of motion of the generating apparatus is essential in the structure of this scheme. We shall say that an event also has the property of motion; the complete description of an event requires a specification of this state of motion"* (pg. 1164).

In our work, we borrow this interpretation of the local time as depending on the state of motion, and consider the absolute time τ as an intrinsic quantity that is set independently for each frame in the sense that it doesn't need a pair of frames to be defined and, in particular, it has the property that each frame registers the same value for the absolute time associated to the occurrence of an event (which is indicated by the equation $\tau = \tau'$).

As a consequence of using (13) into (12) we obtain

$$t + t' = \tau \frac{v(1+a)}{c\sqrt{a^2 - 1}} \tag{26}$$

then, if the absolute time is independent on the state of motion we conclude that in our formalism the relation (26) indicates the local times t, t' are set in connection with the relative state of motion of the frames S, S' without explicit dependence on the position of the event (at least with respect to the combination $t+t'$). As we will see in further sections we will also extend this notion to the dynamical fields assuming that some quantities (the so-called Galilean fields) depends on the state of motion of the observers.

The arbitrariness of the parameter a in the GLT (13) allow us to make some choices for a that may be of particular interest. We analyze some cases.

• Let us assume S' is the rest frame of a particle that moves with velocity $\vec{v}$ relative to another frame S. Let us denote the infinitesimal proper time of the particle as dT, where $dT = dt'$. We obtain from (12) the following relation between the proper time and the absolute time

$$d\tau = \sqrt{a^2 - 1}\frac{c}{v} dT \; . \tag{27}$$

If we conceive the absolute time as a quantity that doesn't depend on the state of motion of the particle then, since the proper time is an intrinsic quantity of the particle that also doesn't depend on the state of motion the particle, we must have

$$\frac{d}{dv}\left(\sqrt{a^2 - 1}\,\frac{c}{v}\right) = 0 \tag{28}$$

which fixes $a = \sqrt{1 + v^2 k^2}$ with k an arbitrary integration constant. If we wish to ensure a dimensionless unit for a we may choose, for example, $k = 1/c$, then

$$a = \sqrt{1 + \frac{v^2}{c^2}} \; . \tag{29}$$

Using this expression for a in (14) we end up with

$$\vec{v} = \gamma_{\tilde{v}}\, \vec{\tilde{v}} \; . \tag{30}$$

If we consider $\vec{\tilde{v}} \to \vec{c}$, then $\gamma_{\tilde{v}} \to \infty$ and consequently we would have $\vec{v} \to \infty$ that implies an instantaneous propagation for the light.

• Another possibility for fixing $a(v)$ is to eliminate the dependence on the relative speed v in equation (26). In fact, from (26) let us introduce a function $f(v) := \frac{v(1+a)}{c\sqrt{a^2-1}}$, where $a \equiv a(v)$. Then, let us impose that $\frac{df}{dv} = 0$, which gives $v\frac{da}{dv} = a^2 - 1$ or

$$a = \frac{1 + k^2 v^2}{1 - k^2 v^2} \tag{31}$$

with $k \in \mathbb{R}$ an arbitrary constant. For this choice of a we have

$$t + t' = \frac{1}{kc}\tau$$

that eliminates any dependence of the local and the absolute time with respect to the relative speed v, which provides an alternative to the interpretation that events depends on the state of motion of the observer.

• A third possibility arises if we impose that $\vec{\widetilde{v}} = \vec{v}$, then equation (14) fixes

$$a(v) = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{32}$$

and with this choice the GLT (13) becomes the ordinary Lorentz transformation

$$\begin{cases} \vec{x}\,' = \vec{x} - (1 - \gamma_v)\frac{1}{v^2}\vec{x}\cdot\vec{v}\vec{v} - \gamma_v t\vec{v} \\ t' = \gamma_v(t - \frac{\vec{x}\cdot\vec{v}}{c^2}) \ . \end{cases} \tag{33}$$

As we see from (29, 31, 32) there are many possible choices for the parameter a, but in all cases it determines the same form for $a(\widetilde{v})$, which results on the same ordinary Lorentz transformation (16) expressed in terms of $\vec{\widetilde{v}}$. In what follows we will consider $a(v)$ as an arbitrary function of v that assumes the same form in terms of $\widetilde{v}$ as given in (15).

4 The Galilei electrodynamics of Lévy-Leblond and Le Bellac

In order to compare our development with the original formalism of the Galilei electrodynamics developed by Levy-Leblond and Le Bellac we review some aspects of [1] where the authors obtained two models for the Galilei electrodynamics depending on the assumptions below (in CGS units):

1. $\widetilde{v}/c \ll 1$, $|\vec{j}| \ll c\rho$, $|\vec{B}| \ll |\vec{E}|$: The electric limit
2. $\widetilde{v}/c \ll 1$, $c\rho \ll |\vec{j}|$, $|\vec{E}| \ll |\vec{B}|$: The magnetic limit

• **The electric limit**

In the case of the electric limit the assumption that $\widetilde{v}/c \ll 1$, $|\vec{j}| \ll c\rho$ gives for the components of the four-current (24, 25) the following transformation

$$\rho' = \rho, \quad \vec{j}\,' = \vec{j} - \rho\vec{\widetilde{v}} \tag{34}$$

and for the electric field transformation (20) we have

$$\begin{aligned}
\vec{E}' &= \frac{1}{\sqrt{1-\frac{\widetilde{v}^2}{c^2}}}\vec{E} + \left(1 - \frac{1}{\sqrt{1-\frac{\widetilde{v}^2}{c^2}}}\right)\frac{1}{\widetilde{v}^2}\,\vec{\widetilde{v}}\cdot\vec{E}\,\vec{\widetilde{v}} + \frac{1}{c}\frac{1}{\sqrt{1-\frac{\widetilde{v}^2}{c^2}}}\,\vec{\widetilde{v}}\times\vec{B} \\
&\simeq \vec{E} + \frac{\vec{\widetilde{v}}}{c}\times\vec{B} \\
&\simeq \vec{E}
\end{aligned} \tag{35}$$

where the last approximation is justified because when $\frac{\widetilde{v}}{c} \ll 1$ and $B \ll E$ we get $\frac{\vec{\widetilde{v}}}{c}\times\vec{B} \ll \vec{E}$.

For the magnetic field transformation (21) we have

$$\begin{aligned}
\vec{B}' &= \frac{1}{\sqrt{1-\frac{\widetilde{v}^2}{c^2}}}\vec{B} + \left(1 - \frac{1}{\sqrt{1-\frac{\widetilde{v}^2}{c^2}}}\right)\frac{1}{\widetilde{v}^2}\,\vec{\widetilde{v}}\cdot\vec{B}\,\vec{\widetilde{v}} - \frac{1}{c}\frac{1}{\sqrt{1-\frac{\widetilde{v}^2}{c^2}}}\vec{\widetilde{v}}\times\vec{E} \\
&\simeq \vec{B} - \frac{\vec{\widetilde{v}}}{c}\times\vec{E}
\end{aligned} \tag{36}$$

where the fact that $\frac{\widetilde{v}}{c} \ll 1$ and $B \ll E$ does not guarantee that $\frac{\vec{\widetilde{v}}}{c}\times\vec{E} \ll \vec{B}$. Now, the Maxwell equation $\vec{\nabla}\times\vec{E} + \frac{1}{c}\partial_t\vec{B} = 0$ indicates that a time varying magnetic field also induces an electric field. Then, a transformation of the type $\vec{E}' = \vec{E}$, tells us that the electric field in both frames coincide, therefore, there is no electric field induced by a time varying magnetic field and this suggests the Maxwell equation $\vec{\nabla}\times\vec{E} + \frac{1}{c}\partial_t\vec{B} = 0$ must be replaced by $\vec{\nabla}\times\vec{E} = 0$, which is imposed by hand. Then, under the assumptions of the electric limit one supposes the following Galilean form for the Maxwell equations

$$\vec{\nabla}\times\vec{E} = 0 \qquad \vec{\nabla}\cdot\vec{B} = 0 \qquad \vec{\nabla}\cdot\vec{E} = 4\pi\rho \qquad \vec{\nabla}\times\vec{B} - \frac{1}{c}\partial_\tau\vec{E} = \frac{4\pi}{c}\vec{j}$$

which is invariant by the transformation of the four-current and the electric and magnetic fields given in (34, 35, 36).

• The magnetic limit

In the case of the magnetic limit the assumption that $\widetilde{v}/c \ll 1$, $c\rho \ll |\vec{j}|$ gives for the components of the four-current (24, 25) the transformation

$$\rho' = \rho - \frac{1}{c^2}\vec{\widetilde{v}}\cdot\vec{j}, \quad \vec{j}\,' = \vec{j}\,. \tag{37}$$

Now, when $E \ll B$ a similar development as the one used in the electric case leads us to the following transformation

$$\begin{aligned} \vec{E}' &= \vec{E} + \frac{1}{c}\vec{\widetilde{v}} \times \vec{B} \\ \vec{B}' &= \vec{B}\ . \end{aligned} \tag{38}$$

Here, it is the fact that $\vec{B}' = \vec{B}$ that forces us to replace the Maxwell equation $\vec{\nabla} \times \vec{B} - \frac{1}{c}\partial_t \vec{E} = \frac{4\pi}{c}\vec{j}$ by $\vec{\nabla} \times \vec{B} = \frac{4\pi}{c}\vec{j}$. The Galilean form of the Maxwell equations in the magnetic limit is then

$$\vec{\nabla} \times \vec{E} + \frac{1}{c}\partial_t \vec{B} = 0 \qquad \vec{\nabla} \cdot \vec{B} = 0 \qquad \vec{\nabla} \cdot \vec{E} = 4\pi\rho \qquad \vec{\nabla} \times \vec{B} = \frac{4\pi}{c}\vec{j}$$

which is invariant under the transformations of the four-current and the fields given in equations (37, 38).

5 The Galilean limit

Special relativity is based on the Lorentz transformation (16). In the limit case when $c \to \infty$ it assumes the form

$$\begin{cases} \vec{x}\,' = \vec{x} - \vec{\widetilde{v}}t \\ t' = t \end{cases} \tag{39}$$

which differs from the form we have written for the Galilei transformation (8) in the role played by the absolute time as time variable, which is not identified with $t = t'$, and also in the difference it exists between $\vec{v}$ and $\vec{\widetilde{v}}$ (see (14)).

In our work, we take the GLT (13) as the basic transformation and we seek the conditions under which the Galilei transformation (8) arises as the limit case. First, we notice that for any arbitrary choice for $a(v)$, equation (14) allows us to express $a(v)$ in terms of $\widetilde{v}$ as

$$a = \frac{1}{\sqrt{1 - \frac{\widetilde{v}^2}{c^2}}} = 1 + \frac{1}{2}\frac{\widetilde{v}^2}{c^2} + \ldots \tag{40}$$

therefore, since $a(\widetilde{v})$ assumes a fixed form as a function on $\widetilde{v}$ it is convenient to set the Galilean limit in terms of a condition on $a(\widetilde{v})$. Then, in the Galilean limit we assume that

$\frac{\widetilde{v}}{c} \ll 1$, or more precisely that $\frac{\widetilde{v}^2}{c^2} \to 0$. In this limit we obtain

$$
\begin{aligned}
t' &= at - \sqrt{a^2-1}\frac{1}{vc}\vec{x}\cdot\vec{v} = \Big(1+\frac{1}{2}\frac{\widetilde{v}^2}{c^2}+\dots\Big)\Big(t-\frac{\widetilde{v}^2}{c^2}\vec{x}\cdot\vec{\widetilde{v}}+\dots\Big) \\
&\simeq t \\
\vec{x}\,' &= \vec{x} - \frac{\Big(-\frac{1}{2}\frac{\widetilde{v}^2}{c^2}+\dots\Big)\Big(1+\frac{1}{2}\frac{\widetilde{v}^2}{c^2}+\dots\Big)}{\big(\frac{\widetilde{v}^2}{c^2}+\dots\big)c^2}\vec{x}\cdot\vec{\widetilde{v}}\,\vec{\widetilde{v}} - \Big(1+\frac{1}{2}\frac{\widetilde{v}^2}{c^2}+\dots\Big)t\vec{\widetilde{v}} \\
&\simeq \vec{x} - \vec{\widetilde{v}}t \\
\tau &= (1-a)\frac{1}{v^2}\vec{x}\cdot\vec{v} + \sqrt{a^2-1}\frac{c}{v}t = \frac{\Big(-\frac{1}{2}\frac{\widetilde{v}^2}{c^2}+\dots\Big)\Big(1+\frac{1}{2}\frac{\widetilde{v}^2}{c^2}+\dots\Big)}{\big(\frac{\widetilde{v}}{c}+\dots\big)c}\frac{1}{v}\,\vec{x}\cdot\vec{\widetilde{v}} + \frac{\widetilde{v}}{v}t \\
&\simeq \frac{\widetilde{v}}{v}t\ .
\end{aligned}
$$

In order to have $\tau = t$ we must assume that in the Galilean limit

$$\widetilde{v} \simeq v\ . \tag{41}$$

6 Defining a natural constant

Let S and S' be two inertial frames moving with relative velocity $\vec{v}$. We wish to define a natural constant associated to each frame, which plays the similar role as the speed of light of the special relativity. Then, let us write down expressions for the light ray velocity as measured by considering derivatives with respect to the absolute time τ. Let $\vec{x}_{S'}$ and $\vec{x}_S$ denote the position of a point in the light front as seen by the frames S, S'. According to the Galilei relativity we write $\vec{x}_{S'} = \vec{x}_S - \vec{v}\tau$. Then,

$$\vec{c}_{S'} = \vec{c}_S - \vec{v} \tag{42}$$

where we have denoted $\vec{c}_S = \frac{d\vec{x}_S}{d\tau}$, and $\vec{c}_{S'} = \frac{d\vec{x}_{S'}}{d\tau}$. But, developing $\frac{d\vec{x}_{S'}}{d\tau} = \frac{d\vec{x}_{S'}}{dt'}\frac{dt'}{d\tau}$ and $\frac{d\vec{x}_S}{d\tau} = \frac{d\vec{x}_S}{dt}\frac{dt}{d\tau}$, and using (12) we obtain

$$\vec{c}_{S'} = \frac{\vec{c}\,'}{\sqrt{a^2-1}\frac{c}{v} - (1-a)\frac{\vec{c}\,'\cdot\vec{v}}{v^2}}\ , \quad \vec{c}_S = \frac{\vec{c}}{\sqrt{a^2-1}\frac{c}{v} + (1-a)\frac{\vec{c}\cdot\vec{v}}{v^2}}\ , \tag{43}$$

and replacing these values in (42) we have

$$\vec{c}\,' = \frac{\vec{c} - \sqrt{a^2-1}\frac{c}{v}\vec{v} - (1-a)\frac{\vec{c}\cdot\vec{v}}{v^2}\vec{v}}{a - \sqrt{a^2-1}\frac{\vec{c}\cdot\vec{v}}{cv}}\ . \tag{44}$$

If we use (14) we are able to rewrite this last expression in terms of $\vec{\widetilde{v}} = \frac{d\vec{x}}{dt}$ and we end up with

$$\vec{c}\,' = \frac{\vec{c} - \gamma\vec{\widetilde{v}} - (1-\gamma)\frac{\vec{c}\cdot\vec{\widetilde{v}}}{\widetilde{v}^2}\vec{\widetilde{v}}}{\gamma\Big(1-\frac{\vec{c}\cdot\vec{\widetilde{v}}}{c^2}\Big)}$$

that is the ordinary velocity transformation for the special relativity. We obtain that $c' = c$, as expected. Taking the Galilean limit (43) we obtain

$$c_{S'} = c_S = c \; . \tag{45}$$

7 Galilean electrodynamics

7.1 The Galilean description of space-time

Given a reference frame S we describe space-time by means of two coordinate systems, which we call the Galilei and the Lorentz systems. These systems are endowed with coordinates that we denote respectively by

$$X_G^\mu \equiv (X_G^0, X_G^i) := (c_S \tau, \vec{x}), \qquad X_L^\mu \equiv (X_L^0, X_L^i) := (ct, \vec{x}) \tag{46}$$

where c_S and c refer to the speed of light defined by considering derivatives with respect to the absolute time or the local time. Both quantities, c_S, c, are considered here as mere factors that allow us to have all coordinates X^μ with the same dimension. In particular, as we have seen in (45), in the Galilean limit we have $c_S = c_{S'} = c$.

The relation between these two systems is established in such a way that given another frame S' endowed with Galilei and Lorentz coordinates denoted by

$$X_G'^\mu \equiv (X_G'^0, X_G'^i) := (c_{S'} \tau, \vec{x}\,'), \qquad X_L'^\mu \equiv (X_L'^0, X_L'^i) := (ct', \vec{x}\,') \tag{47}$$

with S' moving with velocity $\vec{v}$ relative to S we have the following diagram commutative

$$\begin{array}{ccc} (X_G^0, X_G^i) := (c_S \tau, \vec{x}) & \xrightarrow{\;G_{\vec{v}}\;} & (X_G'^0, X_G'^i) := (c_{S'} \tau', \vec{x}\,') \\ \downarrow h & & \downarrow h' \\ (X_L^0, X_L^i) := (ct, \vec{x}) & \xrightarrow{\;L_{\vec{v}}\;} & (X_L'^0, X_L'^i) := (ct', \vec{x}\,') \end{array} \tag{48}$$

where $G_{\vec{v}}$ is the Galilei transformation

$$X_G^\mu \to X_G'^\mu := G_{\vec{v}} X_G^\mu : \begin{cases} \frac{1}{c_{S'}} X_G'^0 = \frac{1}{c_S} X_G^0 \\ \vec{X}_G' = \vec{X}_G - \frac{1}{c_S} X_G^0 \vec{v} \end{cases} \tag{49}$$

and $L_{\vec{v}}$ is the generalized Lorentz transformation (13)

$$X_L^\mu \to X_L'^\mu := L_{\vec{v}} X_L^\mu : \begin{cases} X_L'^0 = a X_L^0 - \sqrt{a^2 - 1}\, \frac{1}{v}\, \vec{X}_L \cdot \vec{v} \\ \vec{X}_L' = \vec{X}_L - (1 - a) \frac{1}{v^2}\, \vec{X}_L \cdot \vec{v}\, \vec{v} - \sqrt{a^2 - 1}\, \frac{1}{v} X_L^0 \vec{v} \; . \end{cases} \tag{50}$$

Here, as we see from the definition of the absolute time (12), for the diagram (48) to be commutative we must set $h := h_{\vec{v}}$, $h' := h_{\vec{v}'} = h_{-\vec{v}}$, with $h_{\vec{v}}, h_{\vec{v}'}$ given by

$$X^\mu_G \to X^\mu_L := h_{\vec{v}} X^\mu_G : \begin{cases} X^0_L = \frac{v}{\sqrt{a_v^2-1}} \left\{ \frac{1}{c_S} X^0_G - (1-a_v) \frac{1}{v^2} \vec{X}_G \cdot \vec{v} \right\} \\ \vec{X}_L = \vec{X}_G \end{cases} \tag{51}$$

$$X'^\mu_G \to X'^\mu_L := h_{\vec{v}'} X'^\mu_G : \begin{cases} X'^0_L = \frac{v'}{\sqrt{a_{v'}^2-1}} \left\{ \frac{1}{c_{S'}} X'^0_G - (1-a_{v'}) \frac{1}{v'^2} \vec{X}'_G \cdot \vec{v}\,' \right\} \\ \vec{X}'_L = \vec{X}'_G \ . \end{cases} \tag{52}$$

We then have

$$h_{-\vec{v}} \circ G_{\vec{v}} = L_{\vec{v}} \circ h_{\vec{v}}$$

or equivalently

$$L_{\vec{v}} = h_{-\vec{v}} \circ G_{\vec{v}} \circ h^{-1}_{\vec{v}} \ . \tag{53}$$

In this form, we are able to see the generalized Lorentz transformation $L_{\vec{v}}$ as induced by the Galilei transformation $G_{\vec{v}}$ through the $h's$ transformations given in (51, 52). We will now establish the transformation properties that arises from the use of one or another of those systems taking as our starting point equation (51).

In the Lorentz system let $\mathcal{F}_{L\mu\nu}$ and $\mathcal{F}^{\mu\nu}_L$ be the electromagnetic field strengths with $\vec{E}$ and $\vec{B}$ given as in (3). In the Galilei system we introduce the corresponding field strengths $F_{G\mu\nu}$, $F^{\mu\nu}_G$ by the relations

$$\mathcal{F}^{\mu\nu}_G = \frac{\partial X^\mu_G}{\partial X^\alpha_L} \frac{\partial X^\nu_G}{\partial X^\beta_L} \mathcal{F}^{\alpha\beta}_L \quad , \qquad \mathcal{F}_{G\mu\nu} = \frac{\partial X^\alpha_L}{\partial X^\mu_G} \frac{\partial X^\beta_L}{\partial X^\nu_G} \mathcal{F}_{L\alpha\beta}$$

whose components are

$$\mathcal{F}^{0i}_G = -\sqrt{a^2-1}\, \frac{c_S}{v}\, E_{Li} + (1-a)\, \frac{c_S}{v^2}\, (\vec{v} \times \vec{B}_L)_i \tag{54}$$

$$\mathcal{F}^{ij}_G = -\epsilon_{ijk} B_{Lk} \tag{55}$$

$$\mathcal{F}_{G0i} = \frac{1}{\sqrt{a^2-1}}\, \frac{v}{c_S}\, E_{Li} \tag{56}$$

$$\mathcal{F}_{Gij} = -\frac{(1-a)}{\sqrt{a^2-1}} \frac{1}{v} (v_i E_{Lj} - v_j E_{Li}) - \epsilon_{ijk} B_{Lk} \ . \tag{57}$$

Contrarily to the Lorentzian case, where the electric and the magnetic fields $\vec{E}$, $\vec{B}$ could be both accommodated (except for a sign) as components of either $(\mathcal{F}^{0i}_L, \mathcal{F}^{ij}_L)$ or $(\mathcal{F}_{L0i}, \mathcal{F}_{Lij})$, here we have $\mathcal{F}^{0i}_G \neq -\mathcal{F}_{G0i}$, $\mathcal{F}^{ij}_G \neq \mathcal{F}_{Gij}$ and this doesn't provide a unique way to identify the Galilean counterpart of the electric and magnetic fields as it was done in (3). In our treatment, is this aspect that determines the existence of the electric and magnetic models for the Galilei electrodynamics.

In the Lorentz system the contravariant four-current is defined as $J_L^\mu \equiv (J_L^0, J_L^i) := (c\rho_L, \vec{j}_L)$, and we denote the corresponding Galilean contravariant four-current as $J_G^\mu \equiv (J_G^0, J_G^i) \equiv (c_S\rho_G, \vec{j}_G)$, which is defined as

$$J_G^\mu := \frac{\partial X_G^\mu}{\partial X_L^\alpha} J_L^\alpha \tag{58}$$

and whose components are

$$\begin{aligned} \rho_G &= \sqrt{a^2-1}\,\frac{c}{v}\rho_L + (1-a)\frac{1}{v^2}\vec{v}\cdot\vec{j}_L \\ \vec{j}_G &= \vec{j}_L \ . \end{aligned} \tag{59}$$

In the same Lorentz system the covariant four-current is defined as $J_{L\mu} \equiv (J_{L0}, J_{Li}) := (c\rho_L, -\vec{j}_L)$, and we denote the corresponding covariant four-current in the Galilei system as $J_{G\mu} \equiv (J_{G0}, J_{Gi}) := (c_S\rho_G, -\vec{j}_G)$ which is defined as

$$J_{G\mu} := \frac{\partial X_L^\alpha}{\partial X_G^\mu} J_{L\alpha} \tag{60}$$

and whose components are

$$\begin{aligned} \rho_G &= \frac{1}{\sqrt{a^2-1}}\,\frac{vc}{c_S^2}\rho_L \\ \vec{j}_G &= \frac{(1-a)}{\sqrt{a^2-1}}\,\frac{c}{v}\rho_L\vec{v} + \vec{j}_L \ . \end{aligned} \tag{61}$$

As we will see, the identification of $\vec{E}$, $\vec{B}$ in terms of the components of the contravariant tensor $\mathcal{F}_G^{\mu\nu}$ and the use of the contravariant four-current J_G^μ will allow us to define the so-called electric limit, while the identification of $\vec{E}$, $\vec{B}$ with the covariant tensor $\mathcal{F}_{G\mu\nu}$ and the use of the covariant four-current $J_{G\mu}$ will produce the magnetic limit.

All these expressions (54, 55, 56, 57, 59, 61) are written relative to the same frame S and relate Galilean quantities $\vec{E}_G, \vec{B}_G, \rho_G, \vec{j}_G$ with the corresponding Lorentzian quantities $\vec{E}_L, \vec{B}_L, \rho_L, \vec{j}_L$ and the velocity $\vec{v}$, a feature that lead us to extend to dynamical quantities such as the Galilean fields the same conjecture made in [11], where the description of events depend on the state of motion of the frame.

In analogy with the commutative diagram (48) relating the coordinates of events, we also have a commutative diagram relating the fields and the four-current as represented schematically below

$$\begin{array}{ccc} (\vec{E}_G, \vec{B}_G, \rho_G, \vec{j}_G, \vec{v}) = \mathcal{K}_G & \xrightarrow{\textit{Galilei}} & \mathcal{K}'_G = (\vec{E}'_G, \vec{B}'_G, \rho'_G, \vec{j}'_G, \vec{v}\,') \\ \downarrow & & \downarrow \\ (\vec{E}_L, \vec{B}_L, \rho_L, \vec{j}_L, \vec{v}) = \mathcal{K}_L & \xrightarrow{\textit{Lorentz}} & \mathcal{K}'_L = (\vec{E}'_L, \vec{B}'_L, \rho'_L, \vec{j}'_L, \vec{v}\,') \end{array}$$

where the vertical arrows are the analogous of the transformations (51, 52) and refer to the transformations connecting the Galilean fields and the Galilean four-current with the corresponding Lorentzian fields and the Lorentzian 4-current.

7.2 The covariant model and the magnetic limit

7.2.1 The covariant model

This model is defined by taking the covariant tensor $\mathcal{F}_{G\mu\nu}$ and the covariant four-current $J_{G\mu}$ as the main element of analysis. Here, we define the electric and magnetic fields as

$$(\vec{E}_G; \vec{B}_G) := (\mathcal{F}_{G0i}; -\frac{1}{2}\epsilon_{ijk}\mathcal{F}_{Gjk}) \ . \tag{62}$$

From (56, 57) we have

$$\begin{aligned} \vec{E}_G &= \frac{1}{\sqrt{a^2-1}} \frac{v}{c_S} \vec{E}_L \\ \vec{B}_G &= \frac{(1-a)}{\sqrt{a^2-1}} \frac{1}{v} \vec{v} \times \vec{E}_L + \vec{B}_L \end{aligned} \tag{63}$$

and the covariant four-current $J_{G\mu} = (c\rho_G, -\vec{j}_G)$ is given by (61).

The transformation of the Galilean fields and the four-current is obtained from

$$\begin{aligned} \mathcal{F}'_{G\mu\nu} &= \frac{\partial X^\alpha_G}{\partial X'^\mu_G} \frac{\partial X^\beta_G}{\partial X'^\nu_G} \mathcal{F}_{G\alpha\beta} \\ J'_{G\mu} &= \frac{\partial X^\alpha_G}{\partial X'^\mu_G} J_{G\alpha} \ , \end{aligned} \tag{64}$$

which gives

$$\begin{cases} \vec{E}'_G = \frac{c_S}{c_{S'}} \vec{E}_G + \frac{1}{c_{S'}} \vec{v} \times \vec{B}_G \\ \vec{B}'_G = \vec{B}_G \end{cases} \tag{65}$$

$$\begin{cases} \rho'_G = \frac{c_S^2}{c_{S'}^2} \rho_G - \frac{1}{c_{S'}^2} \vec{v} \cdot \vec{j}_G \\ \vec{j}\,'_G = \vec{j}_G \ . \end{cases} \tag{66}$$

From (63), expressing $\vec{E}_L, \vec{B}_L$ in terms of $\vec{E}_G$, $\vec{B}_G$, and from (61) expressing ρ_L, $\vec{j}_L$ in terms of $\rho_G, \vec{j}_G$ and replacing them all in (4) we obtain the Maxwell's equations in the

Galilei system

$$
\begin{aligned}
&\vec{\nabla}_G \times \vec{E}_G + \frac{1}{c_S}\partial_\tau \vec{B}_G = 0 \\
&\vec{\nabla}_G \cdot \vec{B}_G = 0 \\
&\vec{\nabla}_G \cdot \vec{E}_G + (1-a)\frac{1}{v^2}\vec{v}\cdot\partial_\tau \vec{E}_G = 4\pi\frac{c_S}{c}\rho_G \\
&\vec{\nabla}_G \times \vec{B}_G + 2(1-a)\frac{c_S}{v^2}\partial_\tau \vec{E}_G + (1-a)\frac{1}{v^2}(\vec{v}\times\partial_\tau \vec{B}_G) + \\
&\qquad +(1-a)\frac{c_S}{v^2}(\vec{v}\cdot\vec{\nabla}_G)\vec{E}_G = \frac{4\pi}{c}\vec{j}_G\ .
\end{aligned}
\tag{67}
$$

The equations shown in (67) assume an awkward form due to the presence of the relative velocity $\vec{v}$ between the frames S, S', which originates from the fact that the transformation h given in (51) (and that was employed to derive relations (67)) is given in terms of $\vec{v}$. This situation is somehow similar to the relation involving $t, \vec{x}$, and τ shown in (12), which also contains the relative velocity. However, the confusion is only apparent if we recall that equation (67) represents the same Maxwell equations, the difference in form arises because they are expressed relative to the Galilei system of coordinates. Here, observing the form of the equations in the Galilei system we can rightly say that the description of classical electrodynamics is simpler and more meaningful when written in the Lorentz system, however, as we will see in the next section the magnetic model of the Galilei electrodynamics arises only when we apply the Galilean limit on the equations (67).

7.2.2 The magnetic limit

The *magnetic limit* consists on taking the Galilean limit shown in section 5 for the expansion of a, $\frac{\tilde{v}}{c} \ll 1$ (which implies $\frac{\tilde{v}^2}{c^2} \to 0$), together with $c\rho_G \ll |\vec{j}_G|$ and $|\vec{E}_G| \ll |\vec{B}_G|$.

In this limit equations (67) become

$$
\begin{aligned}
&\vec{\nabla}_G \times \vec{E}_G + \frac{1}{c}\partial_\tau \vec{B}_G = 0 \\
&\vec{\nabla}_G \cdot \vec{B}_G = 0 \\
&\vec{\nabla}_G \cdot \vec{E}_G = 4\pi\rho_G \\
&\vec{\nabla}_G \times \vec{B}_G = \frac{4\pi}{c}\vec{j}_G
\end{aligned}
\tag{68}
$$

and the corresponding form of the transformations (65, 66) become

$$
\vec{B}'_G = \vec{B}_G \qquad \vec{E}'_G = \vec{E}_G + \frac{1}{c}\vec{v}\times\vec{B}_G \qquad \rho'_G = \rho_G - \frac{1}{c^2}\vec{v}\cdot\vec{j}_G \qquad \vec{j}'_G = \vec{j}_G \tag{69}
$$

where we have used that in the Galilean limit $c_S \to c$ and $\tilde{v} \to v$. The transformation $\rho'_G = \rho_G - \frac{1}{c^2}\vec{v}\cdot\vec{j}_G$ may be justified as follows. We write $\rho'_G = \rho_G\big(1 - \frac{\vec{v}}{c}\cdot\frac{\vec{j}_G}{c\rho_G}\big)$, then we have $\frac{j_G}{c\rho_G} \gg 1$ and $\frac{v}{c} \ll 1$, therefore we cannot neglect $\frac{\vec{v}}{c}\cdot\frac{\vec{j}_G}{c\rho_G}$ in the expression of ρ'_G.

Transformations (69) leave equations (68) invariant and correspond to the same equations obtained in the so-called *magnetic limit* of [1].

7.2.3 The conservation of the four-current

Let us consider the conservation of the four-current $J_{L\mu}$,

$$\partial_t \rho_L + \vec{\nabla}_L \cdot \vec{j}_L = 0$$

which is equivalent to

$$2(a-1)\frac{c_S^2}{v^2}\partial_\tau \rho_G + (1-a)\frac{v_i}{v^2}\partial_\tau j_{Gi} + \partial_{Gi} j_{Gi} - (1-a)\frac{c_S^2}{v^2} v_i \partial_{Gi}\rho_G = 0 \ . \tag{70}$$

In the magnetic limit we have

$$2(a-1)\frac{c_S^2}{v^2}\partial_\tau \rho_G \simeq \partial_\tau \rho_G, \qquad (1-a)\frac{v_i}{v^2}\partial_\tau j_{Gi} \simeq 0, \qquad (1-a)\frac{c_S^2}{v^2} v_i \partial_{Gi}\rho_G \simeq \frac{1}{2} v_i \partial_{Gi}\rho_G$$

where the last term $\frac{1}{2} v_i \partial_{Gi}\rho_G$ may be neglected when compared to the term $\partial_{Gi} j_{Gi}$ since $c_s \rho_G \ll j_G$, therefore we obtain equation (70) as

$$\partial_\tau \rho_G + \vec{\nabla} \cdot \vec{j}_G = 0 \ .$$

7.3 The contravariant model and the electric limit

7.3.1 The contravariant model

This model is defined by taking the contravariant tensor $\mathcal{F}_G^{\mu\nu}$ and the contravariant four-current J_G^μ as the main elements of analysis. We define

$$(\vec{E}_G; \vec{B}_G) := (-\mathcal{F}_G^{0i}; -\frac{1}{2}\epsilon_{ijk}\mathcal{F}_G^{jk}) \ . \tag{71}$$

From (54) and (55) we have

$$\begin{aligned} \vec{E}_G &:= \sqrt{a^2-1}\,\frac{c_S}{v}\,\vec{E}_L - (1-a)\,\frac{c_S}{v^2}\,(\vec{v}\times\vec{B}_L) \\ \vec{B}_G &:= \vec{B}_L \ . \end{aligned} \tag{72}$$

The four-current in the Lorentz system is $J_L^\mu \equiv (J_L^0, J_L^i) := (c\rho_L, \vec{j}_L)$, and we denote the corresponding Galilean contravariant four-current as $J_G^\mu \equiv (J_G^0, J_G^i) \equiv (c\rho_G, \vec{j}_G)$, which is defined as

$$J_G^\mu := \frac{\partial X_G^\mu}{\partial X_L^\alpha} J_L^\alpha$$

and whose components are given by (59).

The transformation of the Galilean fields and the four-current is now obtained from

$$\begin{aligned} \mathcal{F}'^{\mu\nu}_G &= \frac{\partial X'^{\mu}_G}{\partial X^{\alpha}_G}\frac{\partial X'^{\nu}_G}{\partial X^{\beta}_G}\mathcal{F}^{\alpha\beta}_G \\ J'^{\mu}_G &= \frac{\partial X'^{\mu}_G}{\partial X^{\alpha}_G}J^{\alpha}_G \end{aligned} \tag{73}$$

which gives

$$\begin{cases} \vec{E}'_G = \frac{c_{S'}}{c_S}\vec{E}_G \\ \vec{B}'_G = \vec{B}_G - \frac{1}{c_S}\vec{v}\times\vec{E}_G \end{cases} \tag{74}$$

$$\begin{cases} \rho'_G = \rho_G \\ \vec{j}\,'_G = \vec{j}_G - \vec{v}\rho_G\ . \end{cases} \tag{75}$$

In the Galilei system the Maxwell's equations become

$$\begin{aligned} &\vec{\nabla}_G\times\vec{E}_G - 2(1-a)\frac{c_S}{v^2}\partial_\tau\vec{B}_G - (1-a)\frac{c_S}{v^2}(\vec{v}\cdot\vec{\nabla}_G)\vec{B}_G + \\ &+(1-a)\frac{1}{v^2}\vec{v}\times\partial_\tau\vec{E}_G = 0 \\ &\vec{\nabla}_G\cdot\vec{B}_G + (1-a)\frac{1}{v^2}\vec{v}\cdot\partial_\tau\vec{B}_G = 0 \\ &\vec{\nabla}_G\cdot\vec{E}_G = 4\pi\frac{c_S}{c}\rho_G \\ &\vec{\nabla}_G\times\vec{B}_G - \frac{1}{c_S}\partial_\tau\vec{E}_G = \frac{4\pi}{c}\vec{j}_G \end{aligned} \tag{76}$$

7.3.2 The electric limit

The *electric limit* consists on taking the same Galilean limit shown in section 5 for the expansion of a, $\frac{\tilde{v}}{c}\ll 1$ (which implies $\frac{\tilde{v}^2}{c^2}\to 0$), together with $|\vec{j}_G|\ll c\rho_G$ and $|\vec{B}_G|\ll|\vec{E}_G|$.

In this limit equations (76) become

$$\begin{aligned} &\vec{\nabla}_G\times\vec{E}_G = 0 \\ &\vec{\nabla}_G\cdot\vec{B}_G = 0 \\ &\vec{\nabla}_G\cdot\vec{E}_G = 4\pi\rho_G \\ &\vec{\nabla}_G\times\vec{B}_G - \frac{1}{c}\partial_\tau\vec{E}_G = \frac{4\pi}{c}\vec{j}_G \end{aligned} \tag{77}$$

and the transformations of the fields and the four-current become

$$\vec{E}'_G = \vec{E}_G \qquad \vec{B}'_G = \vec{B}_G - \frac{1}{c}\vec{v}\times\vec{E}_G \qquad \rho'_G = \rho_G \qquad \vec{j}'_G = \vec{j}_G - \vec{v}\rho_G\ . \tag{78}$$

It is straightforward to check that equations (77) are left invariant by the transformations (78). Together they correspond to the electric limit of [1].

7.3.3 The conservation of the four-current

Let us consider the conservation of the Lorentzian four-current J_L^μ,

$$\partial_t \rho_G + \vec{\nabla} \cdot \vec{j}_L = 0$$

which is equivalent to

$$\partial_\tau \rho_G + \vec{\nabla}_G \cdot \vec{j}_G = 0 \ .$$

Now, contrarily to the magnetic limit, the conservation of the Galilean four-current follows directly from their counterpart conserved Lorentzian four-current, with no approximation required.

7.4 The general Galilean model

7.4.1 The basic equations

Our third model combines some aspects of the two previous ones. Here, we consider electrodynamics in a medium and introduce the fields $(\vec{E}_G, \vec{B}_G, \vec{D}_G, \vec{H}_G)$ defined in terms of the Galilean fields strengths $\mathcal{F}_{G\mu\nu}$ (56, 57), and $\mathcal{H}_G^{\mu\nu}$

$$\begin{aligned} \mathcal{H}_G^{0i} &= -\sqrt{a^2-1}\,\frac{c_S}{v}\,D_{Li} + (1-a)\,\frac{c_S}{v^2}\,(\vec{v}\times\vec{H}_L)_i \\ \mathcal{H}_G^{ij} &= -\epsilon_{ijk}H_{Lk} \end{aligned} \tag{79}$$

as follows

$$\begin{aligned} E_{Gi} &:= \mathcal{F}_{G0i} \\ B_{Gi} &:= -\frac{1}{2}\epsilon_{ijk}\mathcal{F}_{Gjk} \\ D_{Gi} &:= -\mathcal{H}_G^{0i} \\ H_{Gi} &:= -\frac{1}{2}\epsilon_{ijk}\mathcal{H}_G^{jk} \ . \end{aligned} \tag{80}$$

The four-current in this general model follows the same definition shown in equations (58, 59). We obtain explicitly that

$$\begin{aligned} \vec{E}_G &= \frac{1}{\sqrt{a^2-1}}\,\frac{v}{c_S}\,\vec{E}_L \\ \vec{B}_G &= \vec{B}_L + \frac{(1-a)}{\sqrt{a^2-1}}\frac{1}{v}\,\vec{v}\times\vec{E}_L \\ \vec{D}_G &= \sqrt{a^2-1}\,\frac{c_S}{v}\vec{D}_L - (1-a)\,\frac{c_S}{v^2}\,\vec{v}\,\times\vec{H}_L \\ \vec{H}_G &= \vec{H}_L \\ \rho_G &= \sqrt{a^2-1}\frac{c}{v}\rho_L + (1-a)\frac{1}{v^2}\vec{v}\cdot\vec{j}_L \\ \vec{j}_G &= \vec{j}_L \ . \end{aligned} \tag{81}$$

The transformation of the fields and the four-current follow the same pattern as before and assume the form

$$\begin{cases} \vec{E}'_G = \frac{c_S}{c_{S'}}\vec{E}_G + \frac{1}{c_{S'}}\vec{v} \times \vec{B}_G \\ \vec{B}'_G = \vec{B}_G \\ \vec{D}'_G = \frac{c_{S'}}{c_S}\vec{D}_G \\ \vec{H}'_G = \vec{H}_G - \frac{1}{c_S}\vec{v} \times \vec{D}_G \end{cases} \tag{82}$$

$$\begin{cases} \rho'_G = \rho_G \\ \vec{j}\,'_G = \vec{j}_G - \vec{v}\rho_G \ . \end{cases} \tag{83}$$

In the Galilei system the Maxwell's equations (6) become

$$\begin{aligned} &\vec{\nabla}_G \times \vec{E}_G + \frac{1}{c_S}\partial_\tau \vec{B}_G = 0 \\ &\vec{\nabla}_G \cdot \vec{B}_G = 0 \\ &\vec{\nabla}_G \cdot \vec{D}_G = 4\pi\frac{c_S}{c}\rho_G \\ &\vec{\nabla}_G \times \vec{H}_G - \frac{1}{c_S}\partial_\tau \vec{D}_G = \frac{4\pi}{c}\vec{j}_G \end{aligned} \tag{84}$$

and they assume the same form as the Maxwell's equation in the Lorentz system. Contrarily to what we have seen in the covariant and contravariant models, this form of the Maxwell equations in the Galilei system shown in (84) are invariant under the Galilei transformations of the fields and the four-current (82, 83).

We also notice from the non-homogeneous equations in (84) that the conservation of the Galilean four-current has the form

$$\frac{c_S}{c}\partial_\tau \rho_G + \vec{\nabla}_G \cdot \vec{j}_G = 0 \ . \tag{85}$$

7.4.2 The constitutive relations

Maxwell's equations and the constitutive relations for the Galilean fields can be thought as arising from the formulation of electrodynamics in a Riemannian manifold as follows.

According to [12], [13], in a Riemannian manifold Maxwell's equations assume the form

$$\mathcal{F}_{\alpha\beta;\gamma} + \mathcal{F}_{\beta\gamma;\alpha} + \mathcal{F}_{\gamma\alpha;\beta} = 0, \qquad \mathcal{H}^{\alpha\beta}{}_{;\beta} = -\frac{4\pi}{c}J^\alpha$$

where

$$\mathcal{H}^{\alpha\beta} = g^{\alpha\mu}g^{\beta\nu}\mathcal{F}_{\mu\nu} \tag{86}$$

provides the constitutive relation, which relates the geometry of the spacetime, implicit in the metric $g^{\alpha\beta}$, with the fields present in the electromagnetic tensor $\mathcal{F}_{\alpha\beta}$. Here, if we take spacetime having a Lorentzian metric such that

$$\eta_L^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

we have the corresponding object

$$\eta_G^{\mu\nu} = \frac{\partial X_G^\mu}{\partial X_L^\alpha}\frac{\partial X_G^\nu}{\partial x_L^\beta}\eta_L^{\alpha\beta} = \begin{pmatrix} \frac{2c_S^2}{v^2}(a-1) & \frac{c_S}{v^2}(a-1)v_i \\ \frac{c_S}{v^2}(a-1)v_i & -\delta_{ij} \end{pmatrix} . \tag{87}$$

Assuming $g^{\alpha\beta} = \eta_G^{\alpha\beta}$ in (86) we obtain the constitutive relations for the Galilean fields

$$\begin{aligned} \vec{D}_G &= (a-1)\frac{c_S}{v^2}\vec{v}\times\vec{B}_G + (1-a)^2\frac{c_S^2}{v^4}(\vec{v}\cdot\vec{E}_G)\vec{v} + 2(a-1)\frac{c_S^2}{v^2}\vec{E}_G \\ \vec{H}_G &= \vec{B}_G + (a-1)\frac{c_S}{v^2}\vec{v}\times\vec{E}_G . \end{aligned} \tag{88}$$

Taking the Galilean limit in (88) we obtain

$$\begin{aligned} \vec{D}_G &= \vec{E}_G + \frac{1}{2c}\vec{v}\times\vec{B}_G \\ \vec{H}_G &= \vec{B}_G + \frac{1}{2c}\vec{v}\times\vec{E}_G . \end{aligned}$$

Using the constitutive relations in the non-homogeneous equations of (84) we obtain

$$\begin{aligned} \vec{\nabla}_G\cdot\vec{D}_G = 4\pi\frac{c_S}{c}\rho_G \quad &\longrightarrow \quad \vec{\nabla}_G\cdot\vec{E}_G = 4\pi\rho_{Geq} \\ \vec{\nabla}_G\times\vec{H}_G - \frac{1}{c_S}\partial_\tau\vec{D}_G = \frac{4\pi}{c}\vec{j}_G \quad &\longrightarrow \quad \vec{\nabla}_G\times\vec{B}_G - \frac{1}{c}\partial_\tau\vec{E}_G = \frac{4\pi}{c}\vec{j}_{Geq} \end{aligned} \tag{89}$$

where

$$\begin{aligned} \rho_{Geq} &= \rho_G + \frac{1}{8\pi c}\vec{v}\cdot(\vec{\nabla}_G\times\vec{B}_G) \\ \vec{j}_{Geq} &= \vec{j}_G - \frac{1}{2}\rho_G\vec{v} + \frac{1}{8\pi}(\vec{v}\cdot\vec{\nabla}_G)\vec{E}_G . \end{aligned}$$

Let us use the second non-homogeneous equation given in (89) to write $\vec{\nabla}_G\times\vec{B}_G = \frac{1}{c}\partial_\tau\vec{E}_G + \frac{4\pi}{c}\vec{j}_{Geq}$. Then we may rewrite $\rho_{Geq} = \rho_G + \frac{1}{8\pi c^2}(\vec{v}\cdot\partial_\tau\vec{E} + 4\pi\vec{j}_{Geq})$. Since we are considering the Galilean limit where $1/c^2 \to 0$ we have $\rho_{Geq} = \rho_G$. Then, in the Galilean limit and using the constitutive relations (88) the Maxwell's equations assume the form

$$\begin{cases} \vec{\nabla}_G \times \vec{E}_G + \frac{1}{c}\partial_\tau \vec{B}_G = 0 \\ \vec{\nabla}_G \cdot \vec{B}_G = 0 \\ \vec{\nabla}_G \cdot \vec{E}_G = 4\pi\rho_{Geq} \\ \vec{\nabla}_G \times \vec{B}_G - \frac{1}{c}\partial_\tau \vec{E}_G = \frac{4\pi}{c}\vec{j}_{Geq} \end{cases} \tag{90}$$

$$\begin{cases} \rho_{Geq} = \rho_G \\ \vec{j}_{Geq} = \vec{j}_G - \frac{1}{2}\rho_G\vec{v} + \frac{1}{8\pi}(\vec{v}\cdot\vec{\nabla}_G)\vec{E}_G\ . \end{cases} \tag{91}$$

Here, the conservation of the four-current has the form

$$\partial_\tau \rho_{Geq} + \vec{\nabla}_G \cdot \vec{j}_{Geq} = 0 \tag{92}$$

and since the form of $\vec{j}_{Geq}$ carries explicitly the electric field we must check if this equation (92) doesn't pose any additional restriction on the field. It is straightforward to check that neglecting terms of the order $1/c^2$ we have ensured that

$$\partial_\tau \rho_G + \vec{\nabla}_G \cdot \vec{j}_G = 0 \Rightarrow \partial_\tau \rho_{Geq} + \vec{\nabla}_G \cdot \vec{j}_{Geq} = 0$$

therefore there is no additional condition on the electric and magnetic field. Physically, we assume in this Galilean approximation that we have established the form of j_{Geq} from the medium properties.

Now, in the Galilean limit the transformations of the fields and the four-current reads as

$$\begin{cases} \vec{E}'_G = \vec{E}_G + \frac{1}{c}\vec{v} \times \vec{B}_G \\ \vec{B}'_G = \vec{B}_G \\ \vec{D}'_G = \vec{D}_G \\ \vec{H}'_G = \vec{H}_G - \frac{1}{c}\vec{v} \times \vec{D}_G \end{cases}$$

$$\begin{cases} \rho'_G = \rho_G \\ \vec{j}'_G = \vec{j}_G - \vec{v}\rho_G \end{cases}$$

and it keeps invariant the Galilean form of the Maxwell's equations (90) and the conservation equation (92).

7.5 Another derivation of the constitutive relations

Now, in order to illustrate the validity of the generalized Lorentz transformations for the electromagnetic fields given in (17), (18) let us obtain the same constitutive relations between the fields through the standard procedure. Let us assume the general case of a moving medium at rest relative to the frame S'. Then, we assume the following constitutive relations [8]

$$\vec{D}'_L = \epsilon \vec{E}'_L, \qquad \vec{B}'_L = \mu \vec{H}'_L \ .$$

Using (17) we obtain from $\vec{D}'_L = \epsilon \vec{E}'_L$ that

$$\begin{aligned} & a\vec{D}_L + (1-a)\frac{1}{v^2}\vec{v}\cdot\vec{D}_L\,\vec{v} + \sqrt{a^2-1}\frac{1}{v}\vec{v}\times\vec{H}_L = \\ & = \epsilon\Big\{a\vec{E}_L + (1-a)\frac{1}{v^2}\vec{v}\cdot\vec{E}_L\,\vec{v} + \sqrt{a^2-1}\frac{1}{v}\vec{v}\times\vec{B}_L\Big\} \end{aligned} \tag{93}$$

and taking the scalar product with $\vec{v}$ we get $\vec{v}\cdot\vec{D}_L = \epsilon\,\vec{v}\cdot\vec{E}_L$, which replacing back in (93) let us with

$$a\vec{D}_L + \sqrt{a^2-1}\frac{1}{v}\vec{v}\times\vec{H}_L = \epsilon a\vec{E}_L + \epsilon\sqrt{a^2-1}\frac{1}{v}\vec{v}\times\vec{B}_L \ . \tag{94}$$

Similarly, using (18) we obtain from $\vec{B}'_L = \mu\vec{H}'_L$ that

$$\begin{aligned} & a\vec{B}_L + (1-a)\frac{1}{v^2}\vec{v}\cdot\vec{B}_L\,\vec{v} - \sqrt{a^2-1}\frac{1}{v}\vec{v}\times\vec{E}_L = \\ & = \mu\Big\{a\vec{H}_L + (1-a)\frac{1}{v^2}\vec{v}\cdot\vec{H}_L\,\vec{v} - \sqrt{a^2-1}\frac{1}{v}\vec{v}\times\vec{D}_L\Big\} \end{aligned} \tag{95}$$

and taking the scalar product with $\vec{v}$ it gives $\vec{v}\cdot\vec{B}_L = \mu\,\vec{v}\cdot\vec{H}_L$, which replacing back in (95) gives

$$a\vec{B}_L - \sqrt{a^2-1}\frac{1}{v}\,\vec{v}\times\vec{E}_L = \mu a\vec{H}_L - \mu\sqrt{a^2-1}\frac{1}{v}\vec{v}\times\vec{D}_L \ . \tag{96}$$

Now, taking the vector product with $\vec{v}$ we obtain from (96) that

$$\vec{v}\times\vec{H}_L = \frac{1}{\mu}\vec{v}\times\vec{B}_L + \Big(\epsilon - \frac{1}{\mu}\Big)\frac{\sqrt{a^2-1}}{a}\,\frac{1}{v}(\vec{v}\cdot\vec{E}_L)\vec{v} + \frac{\sqrt{a^2-1}}{a}\,\frac{v}{\mu}\vec{E}_L - \frac{\sqrt{a^2-1}}{a}v\vec{D}_L$$

and replacing this expression for $\vec{v}\times\vec{H}_L$ back in (94) and using $(\vec{v}\cdot\vec{E}_L)\vec{v} = \vec{v}\times(\vec{v}\times\vec{E}_L) + v^2\vec{E}_L$ we get

$$\vec{D}_L = \epsilon\vec{E}_L + \Big(\epsilon - \frac{1}{\mu}\Big)a\sqrt{a^2-1}\frac{1}{v}\vec{v}\times\vec{B}_L - \Big(\epsilon - \frac{1}{\mu}\Big)(a^2-1)\frac{1}{v^2}\vec{v}\times(\vec{v}\times\vec{E}_L) \ . \tag{97}$$

In a similar way, taking the vector product of $\vec{v}$ with (94) we obtain

$$\vec{v}\times\vec{E}_L = \frac{1}{\epsilon}\vec{v}\times\vec{D}_L - \Big(\mu - \frac{1}{\epsilon}\Big)\frac{\sqrt{a^2-1}}{a}\frac{1}{v}(\vec{v}\cdot\vec{H}_L)\vec{v} - \frac{1}{\epsilon}\frac{\sqrt{a^2-1}}{a}v\vec{H}_L + \frac{\sqrt{a^2-1}}{a}v\vec{B}_L$$

and replacing this expression for $\vec{v}\times\vec{E}_L$ into (96) we obtain

$$\mu a\Big(1 - \frac{(a^2-1)}{a^2\mu\epsilon}\Big)\vec{H}_L = \frac{1}{a}\vec{B}_L + \Big(\mu - \frac{1}{\epsilon}\Big)\frac{\sqrt{a^2-1}}{av}\Big\{a\vec{v}\times\vec{D}_L + \frac{\sqrt{a^2-1}}{v}(\vec{v}\cdot\vec{H}_L)\vec{v}\Big\} \ . \tag{98}$$

Again, taking the vector product of $\vec{v}$ with (94) and using that $\vec{v}\times(\vec{v}\times\vec{H}_L)=(\vec{v}\cdot\vec{H}_L)\vec{v}-v^2\vec{H}_L$ we obtain that

$$a\vec{v}\times\vec{D}_L+\sqrt{a^2-1}\frac{1}{v}(\vec{v}\cdot\vec{H}_L)\vec{v}=\sqrt{a^2-1}v\vec{H}_L+\epsilon a\vec{v}\times\vec{E}_L+\epsilon\sqrt{a^2-1}\frac{1}{v}(\vec{v}\cdot\vec{B}_L)\vec{v}-\epsilon\sqrt{a^2-1}v\vec{B}_L$$

that replacing on the rhs of (98) and using $(\vec{v}\cdot\vec{B}_L)\vec{v}=\vec{v}\times(\vec{v}\times\vec{B}_L)+v^2\vec{B}_L$ let us with

$$\vec{H}_L=\frac{1}{\mu}\vec{B}_L+\frac{(\mu\epsilon-1)}{\mu}a\sqrt{a^2-1}\frac{1}{v}\vec{v}\times\vec{E}_L+\frac{(\mu\epsilon-1)}{\mu}(a^2-1)\frac{1}{v^2}\vec{v}\times(\vec{v}\times\vec{B}_L)\,. \tag{99}$$

Finally, using (14) we convert expressions (97, 99) into

$$\begin{aligned}\vec{D}_L &= \epsilon\vec{E}_L+\Big(\epsilon-\frac{1}{\mu}\Big)\frac{\gamma^2}{c}\vec{\tilde{v}}\times\vec{B}_L-\Big(\epsilon-\frac{1}{\mu}\Big)\frac{\gamma^2}{c^2}\vec{\tilde{v}}\times(\vec{\tilde{v}}\times\vec{E}_L)\\ \vec{H}_L &= \frac{1}{\mu}\vec{B}_L+\gamma^2\Big(\epsilon-\frac{1}{\mu}\Big)\left[\frac{\vec{\tilde{v}}}{c}\times\vec{E}_L+\frac{\vec{\tilde{v}}}{c}\times\left(\frac{\vec{\tilde{v}}}{c}\times\vec{B}_L\right)\right]\end{aligned} \tag{100}$$

that are the expected expressions in frame S. Now for a medium at rest relative to frame S and considering the vacuum, we have $\epsilon=1,\mu=1$, and we get the constitutive relations under the form $\vec{D}_L=\vec{E}_L$ and $\vec{H}_L=\vec{B}_L$, which produces the corresponding relations among the Galilean fields

$$\begin{aligned}\vec{D}_G &= (a-1)\frac{c_S}{v^2}\vec{v}\times\vec{B}_G+(1-a)^2\frac{c_S^2}{v^4}(\vec{v}\cdot\vec{E}_G)\vec{v}+2(a-1)\frac{c_S^2}{v^2}\vec{E}_G\\ \vec{H}_G &= \vec{B}_G+(a-1)\frac{c_S}{v^2}\vec{v}\times\vec{E}_G\,,\end{aligned}$$

which agree with the form previously obtained in (88).

8 Conclusion

In our work we obtained the Galilei electrodynamics of Lévy Le Blond and Le Bellac by employing a kind of tensor calculus defined from the relation between two coordinate systems for spacetime, the Galilei and the Lorentz systems, defined in (46). Here, we considered the standard Maxwell electrodynamics as defined relative to the Lorentz system by means of the tensor $\mathcal{F}_{L\mu\nu}$, or $\mathcal{F}_L^{\mu\nu}$. While in the standard Maxwell electrodynamics the relation between the covariant and contravariant components of these tensors corresponds at most to an overall minus sign that doesn't change the form of the Maxwell's equations, we have seen there is a considerable difference when we set the Galilei electrodynamics since the theory that emerges when we take the Galilean limit will depend whether we use the covariant tensor $\mathcal{F}_{G\mu\nu}$ or the contravariant tensor $\mathcal{F}_G^{\mu\nu}$. This makes a clear distinction why there are two Galilean models for the Maxwell electrodynamics, each model arising due to the covariant or contravariant nature of the Galilean electromagnetic tensor we adopt.

The general Galilean model of section 7.4 mixes both constructions as it accommodates the electric and the magnetics fields $\vec{E}_G, \vec{B}_G$ as components of a covariant tensor $\mathcal{F}_{G\mu\nu}$, while the electric and the magnetic excitations $\vec{D}_G, \vec{H}_G$ are accommodated as components of a contravariant tensor $\mathcal{H}^{\mu\nu}_G$. The remarkable aspect of this third model is that it produces the same set of equations for the Galilean and the Lorentzian fields, this time with the absolute time replacing the local time in the Galilei electrodynamics. Also, in this model the form of the constitutive relations we obtained for the Galilei fields shown in section 7.4.2 follows the standard formulation of electrodynamics in a Riemannian manifold, which has agreed with the usual formulation shown in section 7.5. Then, the transformed Lorentzian metric expressed in the Galilei system $\eta^{\mu\nu}_G$ given in (87) plays an essential role. It is still not clear the meaning of this metric $n^{\mu\nu}_G$ in the four-dimensional space with coordinates $X^\mu_G = (x^0_G, \vec{x}_G) = (c_S\tau, \vec{x})$, since it doesn't seem to reveal an euclidean geometry for the space part (x^1_G, x^2_G, x^3_G). Perhaps their limit case when $\frac{1}{c^2} \to 0$ may be of some interest, at least on trying to elucidate the meaning of the speed v that could signalize the existence of a preferred reference frame, or something close to the formulation of an ether as proposed by Dirac [14].

Finally, in the scheme we have presented we still have to analyze the role played by the absolute time τ, and the local time t in establishing the Galilei group as the limit of the Poincaré group when $c \to \infty$.

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