The Quantum Action Principle Revisited

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Abstract

We investigate the basic assumptions leading to Schwinger's quantum action principle in quantum mechanics. We present this principle in a new way that clarifies some previous developments, e.g. the derivation of the fundamental commutators among the canonical variables and the Heisenberg equation for operators. We define operators associated to the classical transformations of the Galilei group, i.e. translations, boosts, and rotations and show their commutators obey the Lie algebra of the Galilei group.

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1 Introduction

Schwinger's quantum action principle (QAP) in quantum mechanics was first presented in [1] and followed the ideas of work originally developed in the context of relativistic quantum field theory [2]. The basic idea of the QAP consists on taking a quantum action $\mathcal{W} = \int_{t_1}^{t_2} dt \, L_q = \int_{t_1}^{t_2} dt \, (p_i \dot{q}_i - H(q, p, t))$, defined in terms of operators, and to consider the boundary term that comes from the variation of the action when we consider infinitesimal variations $\delta_0 q_i := \widetilde{q}_i(t) - q_i(t)$, $\delta_0 p_i := \widetilde{p}_i(t) - p_i(t)$ in the functional form of the canonical operators (examples of these transformations are given in section 5.1 for the cases of translations, Galilean boosts and rotations). For the action given above one obtains as

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boundary term $F = p_i \delta_0 q_i - H \delta t$, which Schwinger interpreted as the generator of unitary transformations corresponding to the freedom of changing the descrition of a quantum mechanical system. Further developments based on this interpretation of F determine the form of the fundamental commutators as well as the Heisenberg equation for operators. Despite its power on fixing several basic properties of quantum mechanical and relativistic field systems, some of the derivations of the QAP are not clear and sometimes invoke assumptions that are not contained in the QAP itself, therefore, leaving the impression the principle is incomplete.

For example, as it is shown in [1], the derivation of the canonical commutators $[q_i, p_j] = i\hbar \delta_{ij}$, $[q_i, q_j] = 0$, $[p_i, p_j] = 0$ involves the use of the QAP twice, at one time taking for action the form given above, and at another time taking for action $\mathcal{W} = \int_{t_1}^{t_2} dt\ L_p = \int_{t_1}^{t_2} dt(\dot{p}_i q_i - H(q, p, t))$. Although the difference between the two lagrangians L_q and L_p relies only on a total derivative, we cannot affirm that both actions will lead to the same physical properties. In fact, this total derivative will appear as a boundary term in the variation of the action giving a contribution to the generator F that may change the description of the system.

Another development that is not clear refers to the derivation of the Heisenberg equation for operators. As the boundary term F is obtained from the infinitesimal variation of an hermitian operator W, then F is both an hermitian and an infinitesimal operator that depends on the infinitesimal variations $\delta_0 q_i, \delta_0 p_i, \delta_0 t$. Therefore, F generates an infinitesimal unitary operator $U = e^{\frac{i}{\hbar} F}$ and its inverse $U^{-1} = e^{-\frac{i}{\hbar} F}$ that, as a first order approximation in $\delta_0 q_i, \delta_0 p_i, \delta_0 t$, write as $U = 1 + \frac{i}{\hbar} F$, $U^{-1} = 1 - \frac{i}{\hbar} F$. The QAP assumes that given an operator $K(q, p, t)$, the infinitesimal unitary operator $U = 1 + \frac{i}{\hbar} F$ induces a change on K as $\delta K = U K U^{-1} - K = -\frac{i}{\hbar}[K, F]$. In order to obtain the Heisenberg equation for K it is assumed additionally in [1] that $\delta K = -(\frac{dK}{dt} - \frac{\partial K}{\partial t})\delta_0 t$, however no proof is given this relation is true in the general case. Another delicate point is the assumption that the operator variations δq_0, δp_0 are functions rather than operators, an issue that led to several investigations [3]. In our approach we are able to let δq_0, δp_0 be operators at the cost of having extra transformations in our theory.

In our work we present a formulation of the QAP that solves the ambiguities mentioned above and follows the general idea behind Schwinger's QAP. The difference is in the choice of the action and in the type of transformations we consider that will render a different form for the boundary term. Another particular characteristic of our development is on the role played by the Hamiltonian that appears independently of the generator F. Both generators will be fundamental in defining unitary transformations in the space of states from which we will be able to determine the fundamental commutators without ambiguity and from a single action. The purpose of this work is to make a comparative study of the

hypothesis that led Schwinger to formulate his QAP and the hypothesis that we use to formulate our version of the QAP. We emphasize throughout the text where we deviate from Schwinger construction and the new features that appear. We hope this may bring a renewed interest on this subject.

Our work is organized as follows. In section **2** we review Schwinger's development of the QAP. Subsection **2.1** is devoted to the basics of unitary transformations in the space of states and observables, a necessary tool to formulate the QAP. Subsection **2.2** reviews Schwinger's derivation of the fundamental commutators and of the Heisenberg equation for operators. In particular, we discuss those aspects which seem incomplete and that demand assumptions besides the QAP itself. In section **3** we formulate a new version of the QAP obtaining a boundary term F that differs from the one obtained by Schwinger. We assume a consistency between composition of transformations generated by F and H, thus obtaining the equations of motion for the canonical operators. Then we analyze the structure of the transformation $\delta_F K$ generated by F on an operator K. In section **4** we analyze the role of the Hamiltonian on the evolution of the system, and we assume it induces on an operator K a transformation $\delta_H K$ having the same structure as the one induced by F. As a result we obtain the Heisenberg equation of motion for K. In section **5** we review symmetry transformations and analyze the conserved quantities arising from Noether theorem. We specialize the transformation to the case of rotation, boost, and translation obtaining the generators of each transformation. Then we show their commutators obey the Lie algebra of the Galilei group.

2 The QAP in Quantum Mechanics

2.1 Transformation theory

[1] Consider two operators $A(t)$, $B(t)$ and their corresponding eigenvectors $|\chi_A, t>$, $|\chi_B, t>$ in the Heisenberg representation. Let us perform unitary transformations on each operator according to

$$
\begin{aligned}
A(t_1) &\longrightarrow \widetilde{A}(t_1) = U_A A(t_1) U_A^{-1} & B(t_2) &\longrightarrow \widetilde{B}(t_2) = U_B B(t_2) U_B^{-1} \\
|\chi_A, t_1> &\longrightarrow |\widetilde{\chi}_A, t_1> := U_A |\chi_A, t_1> & |\chi_B, t_2> &\longrightarrow |\widetilde{\chi}_B, t_2> := U_B |\chi_B, t_2> \ .
\end{aligned}
\tag{1}
$$

The transformation function $<\chi_B, t_2 | \chi_A, t_1>$ changes as

$$
<\widetilde{\chi}_B, t_2 | \widetilde{\chi}_A, t_1> = <\chi_B, t_2 | U_B^{-1} U_A | \chi_A, t_1> \ .
$$

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[1]This follows material presented in [2], section I.

3

In the case of an infinitesimal transformation we write $U_A = 1 + \frac{i}{\hbar}F_A$, $U_B = 1 + \frac{i}{\hbar}F_B$, hence

$$
\begin{aligned}
\delta <\chi_B, t_2|\chi_A, t_1> \; &:= \; <\widetilde{\chi}_B, t_2|\widetilde{\chi}_A, t_1> - <\chi_B, t_2|\chi_A, t_1> \\
&= \; -\frac{i}{\hbar} <\chi_B, t_2|(F_B - F_A)|\chi_A, t_1> \qquad (2) \\
\delta A \; &= \; -\frac{i}{\hbar}[A, F] \; . \qquad (3)
\end{aligned}
$$

Knowledge of the transformation function $<\chi_B, t_2|\chi_A, t_1>$ allows us to determine the dynamical aspects of the physical system. This is due to the fact that all quantities of physical interest are ultimately related to amplitudes of this type. Here, we notice the unitary transformation given in (1) doesn't change the eigenvalue spectra of the operators, however, it does change the transformation function unless we transform both operators by the same unitary transformation, in which case we would have $\delta <\chi_B, t_2|\chi_A, t_1> = 0$. In the general case, equation (2) allows us to relate the effect of a change in the transformation function, caused by an arbitrary change of eigenvectors, as the expectation value of the operator $-\frac{i}{\hbar}(F_B - F_A)$. A particular case of (1) is to consider transformations generated by a family of unitary operators $U(t)$ that, at the instant t, transforms all observables in the same way, i.e. $U_A(t) = U_B(t)$. Equation (2) then writes as

$$
\delta <\chi_B, t_2|\chi_A, t_1> = -\frac{i}{\hbar} <\chi_B, t_2|(F(t_2) - F(t_1))|\chi_A, t_1> \qquad (4)
$$

The essence of the QAP is to find a form for the infinitesimal generators $F(t_2)$, $F(t_1)$ from a dynamical principle.

2.2 The Schwinger QAP in Quantum Mechanics

2.2.1 The Schwinger formulation of the QAP

[2] Let us consider a quantum mechanical system described by canonical variables $\{q_i, p_i\}$ and Hamiltonian $H(q_i, p_i, t)$. We assume it exists a quantum action associated to the system that writes as

$$
W = \int_{t_1}^{t_2} dt \; L(q_i(t), p_i(t), \dot{q}_i(t), \dot{p}_i(t), t) \; . \qquad (5)
$$

In the Heisenberg representation the operators q_i, p_i are time dependent. Therefore, in order to consider independent variations of these quantities and the time t it is convenient to introduce an independent parameter τ such that $t = t(\tau)$, $q_i(\tau) = q_i(t(\tau)), p_i(\tau) = p_i(t(\tau))$.

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[2]This section exhibits the same derivations of [1] but the results are presented in a different way as to facilitate the comparison of Schwinger's development with the one we present in section 3.

Consider now infinitesimal variations in the functional form of these quantities

$$t(\tau) \quad \longrightarrow \quad \widetilde{t}(\tau) = t(\tau) + \delta_0 t(\tau)$$

$$q_i(\tau) \quad \longrightarrow \quad \widetilde{q}_i(\tau) = q_i(\tau) + \delta_0 q_i(\tau) \tag{6}$$

$$p_i(\tau) \quad \longrightarrow \quad \widetilde{p}_i(\tau) = p_i(\tau) + \delta_0 p_i(\tau) \tag{7}$$

where we allow $\delta_0 q_i$, $\delta_0 p_i$ to be operators. In this case we have to set a prescription in order to calculate $\delta K(q, p, \dot{q}, \dot{p})$ for any operator K. We adopt the convention of placing $\delta_0 p$ to the left of $\frac{\partial K}{\partial p}$ and $\delta_0 q$ to the right of $\frac{\partial K}{\partial q}$ (analogously for $\delta_0 \dot{p}$, $\delta_0 \dot{q}$) whenever there is ambiguity in the position of these operators, i.e. $\delta K = \frac{\partial K}{\partial q} \delta_0 q + \delta_0 p \frac{\partial K}{\partial p} + \frac{\partial K}{\partial \dot{q}} \delta_0 \dot{q} + \delta_0 \dot{p} \frac{\partial K}{\partial \dot{p}}$. We notice that

$$\delta_0 \frac{dt}{d\tau} = \frac{d\delta_0 t}{d\tau}, \qquad \delta_0 \frac{d\tau}{dt} = -\frac{d\tau}{dt} \frac{d\delta_0 t}{dt} \tag{8}$$

and

$$\delta_0 \frac{dq_i}{dt} = \delta_0 \left(\frac{d\tau}{dt} \frac{dq_i(\tau)}{d\tau} \right) = -\frac{d\delta_0 t}{dt} \frac{dq_i}{dt} + \frac{d\delta_0 q_i}{dt}$$

$$\delta_0 \frac{dp_i}{dt} = \delta_0 \left(\frac{d\tau}{dt} \frac{dp_i(\tau)}{d\tau} \right) = -\frac{d\delta_0 t}{dt} \frac{dp_i}{dt} + \frac{d\delta_0 p_i}{dt}$$

then we obtain

$$\delta_0 L\left(q_i(\tau), p_i(\tau), \frac{d\tau}{dt} \frac{dq_i(\tau)}{d\tau}, \frac{d\tau}{dt} \frac{dp_i(\tau)}{d\tau}, \tau \right) = \left(\frac{\partial L}{\partial q_i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} \right) \delta_0 q_i + \delta_0 p_i \left(\frac{\partial L}{\partial p_i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{p}_i} \right) +$$

$$+ \left[\frac{\partial L}{\partial t} + \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \dot{q}_i \right) + \frac{d}{dt} \left(\dot{p}_i \frac{\partial L}{\partial \dot{p}_i} \right) \right] \delta_0 t + \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} (\delta_0 q_i - \dot{q}_i \delta_0 t) + (\delta_0 p_i - \dot{p}_i \delta_0 t) \frac{\partial L}{\partial \dot{p}_i} \right). \tag{9}$$

In terms of τ we write

$$W = \int_{\tau_1}^{\tau_2} d\tau \, \frac{dt}{d\tau} \, L\left(q_i(\tau), p_i(\tau), \frac{d\tau}{dt} \frac{dq_i(\tau)}{d\tau}, \tau \right)$$

and using (8, 9) its variation gives

$$\delta W = \int_{t_1}^{t_2} dt \left\{ \left[\frac{\partial L}{\partial t} - \frac{dL}{dt} + \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \dot{q}_i + \dot{p}_i \frac{\partial L}{\partial \dot{p}_i} \right) \right] \delta_0 t + \left(\frac{\partial L}{\partial q_i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} \right) \delta_0 q_i + \right.$$

$$\left. + \delta_0 p_i \left(\frac{\partial L}{\partial q_i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{p}_i} \right) \right\} + \left(L \, \delta_0 t + \frac{\partial L}{\partial \dot{q}_i} (\delta_0 q_i - \dot{q}_i \delta_0 t) + (\delta_0 p_i - \dot{p}_i \delta_0 t) \frac{\partial L}{\partial \dot{p}_i} \right) \Bigg|_{t_1}^{t_2} \tag{10}$$

The QAP imposes that δW depends only on contributions arising from the boundary, i.e.

$$\delta W \equiv F(t_2) - F(t_1) = \left(L \, \delta_0 t + \frac{\partial L}{\partial \dot{q}_i} (\delta_0 q_i - \dot{q}_i \delta_0 t) + (\delta_0 p_i - \dot{p}_i \delta_0 t) \frac{\partial L}{\partial \dot{p}_i} \right) \Bigg|_{t_1}^{t_2}$$

from which we identify the generator of canonical transformations of the system as

$$F(t) := L \, \delta_0 t + \frac{\partial L}{\partial \dot{q}_i} (\delta_0 q_i - \dot{q}_i \delta_0 t) + (\delta_0 p_i - \dot{p}_i \delta_0 t) \frac{\partial L}{\partial \dot{p}_i} \,. \tag{11}$$

The remaining part we impose to be zero. Due to the arbitrariness of the variations $\delta_0 t, \delta_0 q_i, \delta_0 p_i$ we obtain

$$\frac{\partial L}{\partial t} - \frac{dL}{dt} + \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\dot{q}_i + \dot{p}_i\frac{\partial L}{\partial \dot{p}_i}\right) = 0 \tag{12}$$

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = 0, \quad \frac{\partial L}{\partial p_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{p}_i} = 0 \tag{13}$$

Using (13) we note the first equation is in fact equivalent to the operator identity

$$\frac{dL}{dt} = \frac{\partial L}{\partial t} + \frac{\partial L}{\partial q_i}\dot{q}_i + \frac{\partial L}{\partial \dot{q}_i}\ddot{q}_i + \dot{p}_i\frac{\partial L}{\partial p_i} + \ddot{p}_i\frac{\partial L}{\partial \dot{p}_i} \; .$$

Some remarks:

(i) Equation (11) incorporates the form of the generator F for the three quantum lagrangians analysed by Schwinger in [2]: $L_q = p_k\dot{q}_k - H(q_k, p_k, t)$, $L_p = -\dot{p}_k q_k - H(q_k, p_k, t)$ and $L = \frac{1}{2}(L_q + L_p)$.

(ii) It it is possible to work directly with the paremeter t instead of using the parameter τ. In order to obtain (10) we should take the transformations of $t, q_k(t), p_k(t), \dot{q}_k(t), \dot{p}_k(t)$ as

$$t \longrightarrow \tilde{t} = t + \delta_0 t(t)$$
$$q_i(t) \longrightarrow \tilde{q}_i(t) = q_i(t) + \delta_0 q_i(t)$$
$$p_i(t) \longrightarrow \tilde{p}_i(t) = p_i(t) + \delta_0 p_i(t)$$
$$\frac{dq_i(t)}{dt} \longrightarrow \frac{d\tilde{q}_i(t)}{d\tilde{t}} = \frac{dq_i(t)}{dt} + \delta_0\frac{dq_i(t)}{dt}$$
$$\frac{dp_i(t)}{dt} \longrightarrow \frac{d\tilde{p}_i(t)}{d\tilde{t}} = \frac{dp_i(t)}{dt} + \delta_0\frac{dp_i(t)}{dt} \; .$$

The variations $\delta_0\frac{dq_i(t)}{dt}$ and $\delta_0\frac{dp_i(t)}{dt}$ agree with the previous ones, in fact

$$\delta_0\frac{dq_i(t)}{dt} := \frac{d\tilde{q}_i(t)}{d\tilde{t}} - \frac{dq_i(t)}{dt} = \frac{dt}{d\tilde{t}}\frac{d\tilde{q}_i(t)}{dt} - \frac{dq_i(t)}{dt} = \left(1 - \frac{d\delta_0 t}{d\tilde{t}}\right)\frac{d\tilde{q}_i(t)}{dt} - \frac{dq_i(t)}{dt} =$$
$$= \frac{d\delta_0 q_i(t)}{dt} - \frac{d\delta_0 t}{d\tilde{t}}\frac{d\tilde{q}_i(t)}{dt} = \frac{d\delta_0 q_i(t)}{dt} - \frac{d\delta_0 t}{dt}\frac{dq_i(t)}{dt} \tag{14}$$

where the last equality is established as a first order approximation in $\delta_0 t, \delta_0 q_i$. It is important to notice that although we allow for a variation of the parameter t, we keep t fixed when it appears as the argument of $q_i, p_i, \dot{q}_i, \dot{p}_i$. The use of the parameter τ is is just an alternative way to ensure this transformation for $\delta_0\frac{dq_i(t)}{dt}$ and the analogue for $\delta_0\frac{dp_i(t)}{dt}$.

2.2.2 Some derivations from the Schwinger QAP

(1) The Heisenberg equation of motion for operators
Let us choose $L_q = p_i\dot{q}_i - H(q_i, p_i, t)$. From (11) we obtain

$$F(t) = p_i\delta_0 q_i - H\delta_0 t \; . \tag{15}$$

6

The unitary transformation $U = 1 + \frac{i}{\hbar}F(t)$ acts on an operator $K(q_i, p_i, t)$ according to (3),

$$\delta K = -\frac{i}{\hbar}[K, F] = -\frac{i}{\hbar}[K, p_i \delta_0 q_i] + \frac{i}{\hbar}[K, H]\delta_0 t \tag{16}$$

Let us consider $\delta_0 q_i = 0$. Then (15,16) become

$$F = -H\delta_0 t \tag{17}$$

$$\delta K = \frac{i}{\hbar}[K, H]\delta_0 t \tag{18}$$

In this case where $F = -H\delta_0 t$ and in order to obtain the Heisenberg equation for the operator, Schwinger assumes that [3] *δK corresponds to the negative change of K owing to its implicit or dynamical dependence on t*, i.e.

$$\delta K = -\left(\frac{dK}{dt} - \frac{\partial K}{\partial t}\right)\delta_0 t \tag{19}$$

or in a equivalent way

$$\delta K = K(q_i(t - \delta_0 t), p_i(t - \delta_0 t), t) - K(q_i(t), p_i(t), t) \quad \text{[induced by} F = -H\delta_0 t\text{]} \tag{20}$$

This gives the Heisenberg equation

$$\frac{dK}{dt} = \frac{\partial K}{\partial t} - \frac{i}{\hbar}[K, H] \ . \tag{21}$$

Schwinger gives no proof of the general validity of relation (20), henceforth it must be considered as an additional prescription to the QAP.

Remark: Taking $K(q_i(t), p_i(t), t) = q_i(t)$ and $K(q_i(t), p_i(t), t) = p_i(t)$ in (20) we obtain

$$\begin{aligned}
\delta q_i(t) &= q_i(t - \delta_0 t) - q_i(t) = -\dot{q}_i \delta_0 t \ . \\
\delta p_i(t) &= p_i(t - \delta_0 t) - p_i(t) = -\dot{p}_i \delta_0 t
\end{aligned} \tag{22}$$

From the three basic variations associated to the operator $q_i(t)$ [4]

$$\begin{aligned}
\delta_0 q_i(t) &:= \widetilde{q}_i(t) - q_i(t) \ \text{(variation in the functional form)} \\
\overline{\delta} q_i(t) &:= q_i(t + \delta_0 t) - q_i(t) \ \text{(the point variation)} \\
\delta_T q_i(t) &:= \widetilde{q}_i(t + \delta_0 t) - q_i(t) = \delta_0 q_i(t) + \overline{\delta} q_i(t) \ \text{(the total variation)}
\end{aligned}$$

we conclude that (in this case when one takes $F = -H\delta_0 t$) $\delta q_i(t)$ *differs* from $\delta_0 q_i(t)$, $\overline{\delta} q_i(t)$, and $\delta_T q_i(t)$. Therefore, δq_i should be interpreted as a new variation induced by $F = -H\delta_0 t$. The same applies to δp_i.

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[3]Cf. [1], pg 154.

(2) The commutators $[q_i, p_j]$, $[p_i, p_j]$, $[q_i, q_j]$

Let us choose again $L_q = p_j\dot{q}_j - H(q_j, p_j, t)$. Consider $\delta_0 t = 0$. Then (15, 16) become

$$F(t) = p_j \delta_0 q_j$$

$$\delta K = -\frac{i}{\hbar}[K, p_j \delta_0 q_j]$$

At this point, Schwinger obtains part of the fundamental commutators by assuming that $\delta_0 q_j$ *is a function rather than an operator*. This allow us to write

$$\delta K = -\frac{i}{\hbar}[K, p_j]\delta_0 q_j \ . \tag{23}$$

Taking the particular case of $K = q_i$ we obtain

$$\delta q_i = -\frac{i}{\hbar}[q_i, p_j]\delta_0 q_j \tag{24}$$

that is a relation between two unknown quantities: δq_i and the commutator $[q_i, q_j]$. In order to determine the commutator $[q_i, p_j]$ Schwinger assumes that $\delta q_i = \delta_0 q_i$. Then, we have

$$[q_i, p_j] = i\hbar\delta_{ij} \ . \tag{25}$$

By a similar argument, taking $K = p_i$ we obtain

$$[p_i, p_j] = 0 \ . \tag{26}$$

It remains to determine the commutator $[q_i, q_j]$. This time we consider the action as $L_p = -\dot{p}_j q_j - H(q_j, p_j, t)$. From (11) we get

$$F(t) = -\delta_0 p_j\, q_j - H\delta_0 t \ .$$

Taking $\delta_0 t = 0$ and repeating the same development we obtain the commutator $[q_i, q_j] = 0$.

Remark: We assumed $\delta q_i = \delta_0 q_i$. By definition of $\delta_0 q_i$ we obtain $\delta q_i(t) = \widetilde{q}_i(t) - q_i(t)$. Since (24) is a particular case of (23) with $K(q_i(t), p_i(t), t) = q_i(t)$, we have

$$\delta q_i(t) = \widetilde{q}_i(t) - q_i(t) \quad \Rightarrow \quad \delta K(q_i(t), p_i(t), t) \equiv K(\widetilde{q}_i(t), p_i(t), t) - K(q_i(t), p_i(t), t) \tag{27}$$
$$[\text{induced by} F = p_i\,\delta_0 q_i]$$

that establishes the definition of δK induced by $F = p_i\,\delta_0 q_i$. A similar argument applies to $\delta p_i(t)$ which gives

$$\delta K(q_i(t), p_i(t), t) \equiv K(q_i(t), \widetilde{p}_i(t), t) - K(q_i(t), p_i(t), t) \quad [\text{induced by} F = -\delta_0 p_i\, q_i] \tag{28}$$

As we finish this review of Schwinger's QAP, we emphasize the following points:

(i) The variations $\delta_0 q_i, \delta_0 p_i$ are functions rather than operators. This assumption is necessary to establish the commutators. The possibility to have a more general situation force us to consider $\delta_0 q_i, \delta_0 p_i$ as operators.

(ii) The commutators are derived using a particular form for the generator F with $\delta_0 t = 0$. This suggests us to look for an action and transformations of q_k, p_k that results on $F = p_i \delta_0 q_i$ and $F = -\delta_0 p_j\, q_j$ directly from the beginning. The fact of using two different Lagrangians to obtain the commutators indicates that it may be necessary to incorporate the two Lagrangians L_q and L_p into a single action.

(iii) The derivation of the Heisenberg equation for operators assumes a particular form $F = -H \delta_0 t$. This suggests us to consider the Hamiltonian as a separated generator.

(iv) Equations (20, 27, 28) fix the variation δK for different choices of F and. These cases assume that $\delta_0 q_i, \delta_0 p_i$ weren't operators. What will happen to δK if we allow $\delta_0 q_i, \delta_0 p_i$ to be operators?

3 A new formulation of the QAP in quantum mechanics

3.1 Establishing the QAP

In this section we intend to present the QAP in a way that incorporates the observations made in (i)-(iv). Again, let us consider a quantum mechanical system described by canonical variables $\{q_i, p_i\}$ with Hamiltonian $H(q_i, p_i, t)$. In order to keep the discussion as simple as possible and to avoid non-essential complications due to spin we assume that all observables can be written in terms of these fundamental operators. Let us suppose that the initial and final configurations of the system happen at instants t_1, t_2. We state the QAP as follows: *There exists a functional of the canonical variables defined by*

$$W \equiv \int_{t_1}^{t_2} dt \left(p_i \dot{q}_i - \dot{p}_i q_i - 2H(q, p, t) \right) \tag{29}$$

such that under a transformation of the canonical variables ($\delta_0 q_i, \delta_0 p_i$ being operators)

$$\begin{aligned} q_i(t) &\longrightarrow \widetilde{q}_i(t) := q_i(t) + \delta_0 q_i(t) \\ p_i(t) &\longrightarrow \widetilde{p}_i(t) := p_i(t) + \delta_0 p_i(t) \end{aligned} \tag{30}$$

it transforms as $\delta W = F(t_2) - F(t_1)$. This quantity is related to the variation of the amplitude $<\chi_B, t_2|\chi_A, t_1>$ by

$$\delta <\chi_B, t_2|\chi_A, t_1> := -\frac{i}{\hbar} <\chi_B, t_2|\delta W|\chi_A, t_1> \tag{31}$$

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where $F(t_1)$ and $F(t_2)$ are generators of unitary transformations acting on the states $|\chi_A, t_1 >$ and $|\chi_B, t_2 >$.

From (30) we have

$$\delta W = \left(p_i\, \delta_0 q_i - \delta_0 p_i\, q_i \right)\Big|_{t_1}^{t_2} + 2\int_{t_1}^{t_2} dt\left(-\dot{p}_i\, \delta_0 q_i + \delta_0 p_i\, \dot{q}_i - \frac{\partial H}{\partial q_i}\delta_0 q_i - \delta_0 p_i\frac{\partial H}{\partial p_i} \right) .$$

Assuming the QAP we must have $\delta W = F(t_2) - F(t_1)$. Therefore, we identify

$$F(t) \quad = \quad p_i\, \delta_0 q_i - \delta_0 p_i\, q_i \Big|_t . \tag{32}$$

The QAP fixes the remaining part as zero i.e.

$$\int_{t_1}^{t_2} dt\left(-\dot{p}_i\, \delta_0 q_i + \delta_0 p_i\, \dot{q}_i - \frac{\partial H}{\partial q_i}\delta_0 q_i - \delta_0 p_i\frac{\partial H}{\partial p_i} \right) = 0$$

and since the variations $\delta_0 q_i$, $\delta_0 p_i$ are arbitrary we must have

$$\frac{\partial H}{\partial q_i} = -\dot{p}_i, \quad \frac{\partial H}{\partial p_i} = \dot{q}_i . \tag{33}$$

Consider now

$$H(q, +\delta_0 q, p + \delta_0 p, t + \delta t) - H(q, p, t) = \frac{\partial H}{\partial q_i}\delta_0 q_i + \delta_0 p_i\frac{\partial H}{\partial p_i} + \frac{\partial H}{\partial t}\delta t.$$

Taking the particular case of variations $\delta_0 q_i = \dot{q}_i \delta t$, $\delta_0 p_i = \dot{p}_i \delta t$ and using (33) we have

$$\frac{dH}{dt} = \frac{\partial H}{\partial t} . \tag{34}$$

The boundary term we obtained in (32) differs from the one derived by Schwinger, e.g. $F = p_i \delta_0 q_i - H\delta_0 t$, in that we do not have any contribution arising from the Hamiltonian. The reason is that we do not consider time transformations in (30). This separation of the Hamiltonian from the generator F suggest the time evolution of states is generated by H alone.

Now, we should check the consistency between unitary transformations generated by F and H. Let us consider a time independent Hamiltonian (i.e. $\frac{\partial H}{\partial t} = 0$) and the following transformations

$$|\chi, t > \xrightarrow{e^{\frac{i}{\hbar}F(t)}} |\widetilde{\chi}, t >= e^{\frac{i}{\hbar}F(t)}|\chi, t >, \quad |\chi, t_0 > \xrightarrow{e^{\frac{i}{\hbar}H(t-t_0)}} |\chi, t >= e^{\frac{i}{\hbar}H(t-t_0)}|\chi, t_0 > .$$

Consistency requires that

$$
\begin{array}{ccccc}
|\chi, t_0 > & \xrightarrow{e^{\frac{i}{\hbar}H(t-t_0)}} & |\chi, t > & \xrightarrow{e^{\frac{i}{\hbar}F(t)}} & |\widetilde{\chi}, t > \\
|\chi, t_0 > & \xrightarrow{e^{\frac{i}{\hbar}F(t_0)}} & |\widetilde{\chi}, t_0 > & \xrightarrow{e^{\frac{i}{\hbar}H(t-t_0)}} & |\widetilde{\chi}, t >
\end{array}
$$

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i.e.

$$e^{\frac{i}{\hbar}F(t)} = e^{\frac{i}{\hbar}H(t-t_0)} e^{\frac{i}{\hbar}F(t_0)} e^{-\frac{i}{\hbar}H(t-t_0)} \ .$$

For an infinitesimal transformation with $\delta t = t - t_0$ this gives $F(t) = F(t_0) + \frac{i}{\hbar} \delta t [H, F(t_0)]$, or equivalently

$$\frac{dF}{dt} = -\frac{i}{\hbar}[F, H] \ .$$

Using (32), the previous equation can be rewritten as

$$\dot{p}_i\,\delta_0 q_i + p_i\,\delta_0 \dot{q}_i - \delta_0 \dot{p}_i\,q_i - \delta_0 p_i\,\dot{q}_i = -\frac{i}{\hbar}\,[p_i, H]\delta_0 q_i - \frac{i}{\hbar}\,p_i[\delta_0 q_i, H] + \frac{i}{\hbar}\,\delta_0 p_i[q_i, H] + \frac{i}{\hbar}\,[\delta_0 p_i, H]q_i$$

which gives

$$
\begin{aligned}
\dot{q}_i &= -\tfrac{i}{\hbar}[q_i, H], & \dot{p}_i &= -\tfrac{i}{\hbar}[p_i, H] \\
\delta_0\,\dot{q}_i &= -\tfrac{i}{\hbar}[\delta_0 q_i, H], & \delta_0\,\dot{p}_i &= -\tfrac{i}{\hbar}[\delta_0 p_i, H] \ .
\end{aligned}
\tag{35}
$$

The commutators involving the variations $\delta_0 q_i$, $\delta_0 p_i$ are in general different from zero, a fact that is only possible by assuming them to be operators. Another condition arises when we consider $\dot{q}_i = -\frac{i}{\hbar}[q_i, H]$ and take the variation δ_0 directly from this relation: $\delta_0 \dot{q}_i = -\frac{i}{\hbar}[\delta_0 q_i, H] - \frac{i}{\hbar}[q_i, \delta_0 H]$, that gives

$$[q_i, \delta_0 H] = 0 \ . \tag{36}$$

Analogously we obtain

$$[p_i, \delta_0 H] = 0 \ . \tag{37}$$

These equations are consistency conditions to be satisfied by the variations $\delta_0 q_i$, $\delta_0 p_i$. It becomes clear that the choice of the Hamiltonian restrict the type of transformations (30) we can consider for the system.

In Schwinger's original formulation [1], the variations $\delta_0 q_i$, $\delta_0 p_i$ were assumed to commute with the canonical variables. Therefore, the equations he obtains is $\delta_0 \dot{q}_i = -\frac{i}{\hbar}[q_i, \delta_0 H]$, $\delta_0 \dot{p}_i = -\frac{i}{\hbar}[p_i, \delta_0 H]$ which differs from ours. Consistency conditions (36, 37) are absent in Schwinger's formulation and appear here due to the operator character of the variations $\delta_0 q$, $\delta_0 p$.

3.2 The fundamental commutators

We have not determined yet any fundamental commutators among the canonical variables, therefore, the previous equations do not establish any dynamical aspect of the model.

We assume the generator F induce an infinitesimal transformation on the canonical variables $q_i(t), p_i(t)$ that we write as $\delta_F q_i \equiv \bar{\delta}_F q_i + \delta_0 q_i$, $\delta_F p_i \equiv \bar{\delta}_F p_i + \delta_0 p_i$. As we will see below, the additional terms $\bar{\delta}_F q_i, \bar{\delta}_F p_i$ are required in order to garantee the consistency of the formalism in the case $\delta_0 q_i, \delta_0 p_i$ are operators.

In order to fix the fundamental commutators, let us consider the action of the generator F on an operator $K(q, p, t)$. Since $F = p_i \, \delta_0 q_i - \delta_0 p_i \, q_i$ doesn't depend on t, it is reasonable to assume from (27, 28) that

$$\delta_F K(q, p, t) := \widetilde{K}(q + \delta_0 q, p + \delta_0 p, t) - K(q, p, t) \equiv \bar{\delta}_F K + \delta_0 K \qquad (38)$$

where

$$\delta_0 K := \frac{\partial K}{\partial q_i} \, \delta_0 q_i + \delta_0 p_i \frac{\partial K}{\partial p_i} \qquad (39)$$

$$\bar{\delta}_F K := \widetilde{K}(q, p, t) - K(q, p, t) \qquad (40)$$

with $\bar{\delta}_F K$ referring to an arbitrary change in the functional form of the operator K. Also,

$$\delta_F K \equiv U K U^{-1} - K = -\frac{i}{\hbar}[K, F] \, . \qquad (41)$$

From (38,41) we obtain the following equation

$$\bar{\delta}_F K + \frac{\partial K}{\partial q_i} \, \delta_0 q_i + \delta_0 p_i \frac{\partial K}{\partial p_i} = -\frac{i}{\hbar}[K, p_i]\delta_0 q_i - \frac{i}{\hbar} \, p_i[K, \delta_0 q_i] + \frac{i}{\hbar} \, \delta_0 p_i[K, q_i] + \frac{i}{\hbar} \, [K, \delta_0 p_i]q_i$$

which gives

$$\frac{\partial K}{\partial q_i} = -\frac{i}{\hbar}[K, p_i], \quad \frac{\partial K}{\partial p_i} = \frac{i}{\hbar}[K, q_i] \qquad (42)$$

$$\bar{\delta}_F K = \frac{i}{\hbar} \, [K, \delta_0 p_i] \, q_i - \frac{i}{\hbar} \, p_i \, [K, \delta_0 q_i] \, . \qquad (43)$$

We see from (43) that $\bar{\delta}_F K$ *is determined by the commutators of K with the operators $\delta_0 q_i$, $\delta_0 p_i$.* In particular, taking for K the q_k and p_k in (41) we obtain the commutator relations

$$[q_i, p_j] = i\hbar \, \delta_{ij}, \quad [q_i, q_j] = [p_i, p_j] = 0 \qquad (44)$$

and $\bar{\delta}_F q_i = \frac{i}{\hbar} \, [q_i, \delta_0 p_j] \, q_j - \frac{i}{\hbar} \, p_j \, [q_i, \delta_0 q_j]$, $\quad \bar{\delta}_F p_i = \frac{i}{\hbar} \, [p_i, \delta_0 p_j] \, q_j - \frac{i}{\hbar} \, p_j \, [p_i, \delta_0 q_j]$. Placing K as $\delta_0 q$, $\delta_0 p$ into (42) we obtain

$$\bar{\delta}_F q_i = -\frac{\partial \delta_0 p_j}{\partial p_i} q_j + p_j \frac{\partial \delta_0 q_j}{\partial p_i}, \quad \bar{\delta}_F p_i = \frac{\partial \delta_0 p_j}{\partial q_i} q_j - p_j \frac{\partial \delta_0 q_j}{\partial q_i} \qquad (45)$$

that works as an alternative definition for $\overline{\delta}_F q_i$, $\overline{\delta}_F p_i$. From (32) we write [4]

$$
\begin{aligned}
\overline{\delta}_F q_i = \frac{dF}{dp_i} - \delta_0 q_i &\implies \delta_F q_i = \frac{dF}{dp_i} \\
\overline{\delta}_F p_i = -\frac{dF}{dq_i} - \delta_0 p_i &\implies \delta_F p_i = -\frac{dF}{dq_i}
\end{aligned}
\tag{46}
$$

which resembles equations (33) with H in place of F. It should be noticed that the effect of the generator F on the canonical variables is to produce a change $\delta_F q = \overline{\delta}_F q + \delta_0 q$, $\delta_F p = \overline{\delta}_F p + \delta_0 p$, that adds contributions $\overline{\delta}_F q$, $\overline{\delta}_F p$ to the original arbitrary variations $\delta_0 q$, $\delta_0 p$. Equation (38) is an assumption we have to make in order to calculate the fundamental commutators. It gives the response of $K(q, p, t)$ to the action of the generator F. It has the same role as the assumption made by Schwinger that $\delta K = -(\frac{dK}{dt} - \frac{\partial K}{\partial t})\delta t$, although the contents of one and another are quite different.

4 The Dynamical Evolution of the System and the Hamiltonian

4.1 The Heisenberg equation of motion for an operator

We consider now the role of the Hamiltonian on the dynamics of the system. Let us extend the previous construction considering $F = H\delta t$, i.e. we assume the generator $H\delta t$ induce an infinitesimal transformation on the canonical variables that we write as $\delta_H q = \overline{\delta}_H q + \delta_t q$, $\delta_H p = \overline{\delta}_H p + \delta_t p$. The form of this transformation will be fixed below.

Given an operator K, in analogy with (41), we assume it transforms under the action of the generator H as

$$
\delta_H K = -\frac{i}{\hbar}[K, H\delta t]
\tag{47}
$$

with (20) suggesting us to write $\delta_H K = \widetilde{K}(q + \delta_t q, p + \delta_t p, t) - K(q, p, t) \equiv \overline{\delta}_H K + \delta_t K$ with $\overline{\delta}_H K$ and $\delta_t K$ defined as in (39,40)

$$
\delta_t K := \frac{\partial K}{\partial q_i}\delta_t q_i + \delta_t p_i \frac{\partial K}{\partial p_i}
\tag{48}
$$

$$
\overline{\delta}_H K := \widetilde{K}(q, p, t) - K(q, p, t) .
\tag{49}
$$

Thus we obtain [5]

$$
\overline{\delta}_H K(q, p, t) + \frac{\partial K}{\partial q_i}\delta_t q_i + \delta_t p_i \frac{\partial K}{\partial p_i} = -\frac{i}{\hbar}[K, H]\delta t - \frac{i}{\hbar}H[K, \delta t]
$$

---

[4]We denoted $\frac{dF}{dq_i} \doteq \frac{\partial F}{\partial q_i} + \frac{\partial F}{\partial \delta_0 q_j}\frac{\partial \delta_0 q_j}{\partial q_i} + \frac{\partial \delta_0 p_j}{\partial q_i}\frac{\partial F}{\partial \delta_0 p_j}$, $\quad \frac{dF}{dp_i} \doteq \frac{\partial F}{\partial p_i} + \frac{\partial F}{\partial \delta_0 q_j}\frac{\partial \delta_0 q_j}{\partial p_i} + \frac{\partial \delta_0 p_j}{\partial p_i}\frac{\partial F}{\partial \delta_0 p_j}$.

[5]We use the same idea explicit in eq. (43) of taking $\overline{\delta}_H K$ associated to the commutator of K with the parameter δt.

or

$$\frac{\partial K}{\partial q_i}\delta_t q_i + \delta_t p_i \frac{\partial K}{\partial p_i} = -\frac{i}{\hbar}[K,H]\delta t \tag{50}$$

$$\overline{\delta}_H K(q,p,t) = -\frac{i}{\hbar}H[K,\delta t]$$

Now, since the variation δt is not an operator we obtain $\overline{\delta}_H K = \overline{\delta}_H q = \overline{\delta}_H p = 0$. Taking $K = q$ in (47) and using (35) we have $\delta_t q_i = -\frac{i}{\hbar}[q_i,H]\delta t = \dot{q}_i \delta t$. Analogously, $\delta_t p_i = \dot{p}_i \delta t$. Replacing these values into (50) we obtain the Heisenberg equation for the operator

$$\frac{dK}{dt} = \frac{\partial K}{\partial t} - \frac{i}{\hbar}[K,H] . \tag{51}$$

A final consistency check consists to put $K = H$ into (47), which gives $\delta_t H = -\frac{i}{\hbar}[H,H]\delta t = 0$. Also, by definition we have $\delta_t H := \frac{\partial H}{\partial q_i}\delta_t q_i + \delta_t p_i \frac{\partial H}{\partial p_i} = -\dot{p}_i \dot{q}_i \delta t + \dot{p}_i \delta t \dot{q}_i = 0$ upon using (33). Finally, we notice that

$$\delta_H q_i = \overline{\delta}_H q_i + \delta_t q_i = \dot{q}_i \delta t = \frac{\partial H}{\partial p_i}\delta t$$

$$\delta_H p_i = \overline{\delta}_H p_i + \delta_t p_i = \dot{p}_i \delta t = -\frac{\partial H}{\partial q_i}\delta t$$

that is consistent with (46) upon the identification $F_t = H\delta t$.

4.2   The Heisenberg equation for the q-eigenstates

As an application of the previous development let us consider the position eigenstates $|q,t>$. Considering $F_t = H\delta t$ we have $\delta_H|q,t> = \frac{i}{\hbar}\delta t H|q,t>$. We develop $\delta_H|q,t>$ as [6]

$$\delta_H|q,t> := |q,t+\delta t> - |q,t> = \delta t \frac{\partial}{\partial t}|q,t> .$$

Then we get

$$\frac{\partial}{\partial t}|q,t> = \frac{i}{\hbar}H|q,t> \tag{52}$$

In Schwinger's approach [1], equation (52) is obtained in a quite different manner, as it follows from $F = p_i \delta_0 q_i - H\delta t$ and $\delta|q,t> = -\frac{i}{\hbar}F|q,t> = \frac{i}{\hbar}(p_i \delta_0 q_i - H\delta t)|q,t>$. Then, if $\delta_0 q_i$ is a c-number, together with δt, in Schwinger's formalism one can formally calculate the functional derivatives $\frac{\delta}{\delta q_i}|q,t>$, $\frac{\delta}{\delta t}|q,t>$, this last one we associate to equation (52).

---

[6]Since the transformation is unitary it doesn't affect the eigenvalues, therefore the only contribution is due to the time variation.

5 Symmetry Transformations and Conserved Quantities

5.1 Noether Theorem

We analyze now symmetry transformations and the associated conserved quantities. We will follow the same development as [4] adapted to an Hamiltonian formalism. Let us denote $w = p_i \dot{q}_i - \dot{p}_i q_i - 2H(q,p)$ and take

$$W[q(t), p(t), \dot{q}(t), \dot{p}(t), t] = \int dt \, w(q(t), p(t), \dot{q}(t), \dot{p}(t)) \, .$$

We submit the canonical variables and the time to arbitrary transformations of the type

$$
\begin{aligned}
t &\rightarrow t' = t + \delta t \\
q_i &\rightarrow q_i'(t') = q_i(t) + \delta q_i(t) \\
p_i &\rightarrow p_i'(t') = p_i(t) + \delta p_i(t)
\end{aligned}
\tag{53}
$$

that are related to $\delta_0 q_i$, $\delta_0 p_i$ by

$$\delta q_i(t) = \delta_0 q_i(t) + \dot{q}_i(t)\delta t, \quad \delta p_i(t) = \delta_0 p_i(t) + \dot{p}_i(t)\delta t \, .$$

Another useful relation is

$$\delta \frac{dq_i}{dt} = \frac{d\delta q_i}{dt} - \frac{d\delta t}{dt}\frac{dq_i}{dt}, \quad \delta \frac{dp_i}{dt} = \frac{d\delta p_i}{dt} - \frac{d\delta t}{dt}\frac{dp_i}{dt} \, .$$

In addition to transformations (53) we also consider an arbitrary change on the functional form of $w(q,p)$ that is independent on the functional changes of the canonical variables and keeps the equations of motion invariant, e.g. $w(q,p) \rightarrow w(q,p) + \widetilde{\Omega}(q,p)$. The only possibility comes from a functional change on the Hamiltonian, $H \rightarrow H' = H + \Omega(q,p)$ ($\widetilde{\Omega} \equiv -2\Omega$) . In terms of this modified Hamiltonian H' the equations of motion write as $\dot{q}_i = \frac{i}{\hbar}[q_i, H'] = \frac{i}{\hbar}[q_i, H + \Omega]$, $\dot{p}_i = \frac{i}{\hbar}[p_i, H'] = \frac{i}{\hbar}[p_i, H + \Omega]$ which stay invariant if $[q_i, \Omega(q,p)] = [p_i, \Omega(q,p)] = 0$, that gives $\Omega(q,p)$ constant, which we can ignore. Transformations (53) are a *symmetry transformation* [4] if

$$W'[q'(t'), p'(t'), \dot{q}'(t'), \dot{p}'(t'), t'] = W[q(t), p(t), \dot{q}(t), \dot{p}(t)]$$

which gives

$$\int dt \left\{ \frac{d}{dt}\left(p_i \, \delta q_i - \delta p_i \, q_i - 2H\delta t \right) - 2\left(\dot{p}_i + \frac{\partial H}{\partial q_i} \right)\delta q_i + 2\delta p_i\left(\dot{q}_i - \frac{\partial H}{\partial p_i} \right) + 2\left(\frac{dH}{dt} - \frac{\partial H}{\partial t} \right)\delta t \right\} = 0$$

Using the equations of motion we obtain

$$\frac{d}{dt}\left(p_i \, \delta q_i - \delta p_i \, q_i - 2H\delta t \right) = 0$$

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and from this we write the corresponding conserved quantity

$$Q(t) = p_i\,\delta q_i - \delta p_i\,q_i - 2H\delta t \;.\qquad(54)$$

Let us consider transformations of the canonical variables that corresponds to infinitesimal translations, rotations and boosts. Their effect on the canonical variables is

$$q_i \;\to\; q_i' = q_i + \delta a_i + (\delta\omega \times q)_i + \delta v_i t$$
$$p_i \;\to\; p_i' = p_i + (\delta\omega \times p)_i + m\delta v_i$$

Let us consider the particular cases:

(i) Boosts: $\delta q_i := \delta v_i t$, $\delta p_i := m\delta v_i$. The conserved current is $Q(t) = (p_i t - mq_i)\delta v_i$ and the boost generators are identified as $N_i := p_i t - mq_i$ with $\frac{dQ(t)}{dt} = 0 \Rightarrow \frac{dN_i}{dt} = 0$.

(ii) Rotations: $\delta q_i := \epsilon_{ijk}\delta\omega_j\,q_k$, $\delta p_i := \epsilon_{ijk}\delta\omega_j\,p_k$. The conserved current is $Q(t) = 2\epsilon_{ijk}q_j p_k\delta\omega_i$ and the rotation generators are $J_i = \epsilon_{ijk}q_j p_k$ with $\frac{dJ_i}{dt} = 0$.

(iii) Translations: $\delta q_i := \delta a_i$, $\delta p_i := 0$. The conserved current is $Q(t) = p_i\delta a_i$ and the translation generators are $P_i = p_i$ with $\frac{dP_i}{dt} = 0$.

5.2   Galilei algebra

We analyze now the commutators between the generators $\vec{P}$, $\vec{J}$, $\vec{N}$, H for the case of a massive spinless particle. Using the Heisenberg equation for operators (51) we obtain

$$\begin{aligned}
[P_i, H] &= i\hbar\left(\frac{d}{dt} - \frac{\partial}{\partial t}\right)P_i = -i\hbar\frac{\partial P_i}{\partial t} = -i\hbar\frac{\partial p_i}{\partial t} = 0\\
[J_i, H] &= i\hbar\left(\frac{d}{dt} - \frac{\partial}{\partial t}\right)J_i = -i\hbar\frac{\partial J_i}{\partial t} = -i\hbar\frac{\partial(\epsilon_{ijk}q_j p_k)}{\partial t} = 0\\
[N_i, H] &= i\hbar\left(\frac{d}{dt} - \frac{\partial}{\partial t}\right)N_i = -i\hbar\frac{\partial N_i}{\partial t} = -i\hbar\frac{\partial(p_i t - mq_i)}{\partial t} = -i\hbar p_i = -i\hbar P_i \;.
\end{aligned}$$

The other relations follow from the fundamental commutators (44)

$$\begin{aligned}
[P_i, P_j] &= [p_i, pj] = 0\\
[J_i, P_j] &= [\epsilon_{ikl}q_k p_l, p_j] = \epsilon_{ikl}[q_k, p_j]p_l = i\hbar\epsilon_{ijl}p_l = i\hbar\epsilon_{ijl}P_l\\
[N_i, P_j] &= [p_i t - mq_i, p_j] = -i\hbar\,m\delta_{ij}\\
[J_i, J_j] &= i\hbar\epsilon_{ijk}J_k\\
[N_i, J_j] &= [p_i t - mq_i, \epsilon_{jkl}q_k p_l] = \epsilon_{jkl}t[p_i, q_k]p_l - m\epsilon_{jkl}q_k[q_i, p_l] = i\hbar\epsilon_{ijk}N_k\\
[N_i, N_j] &= [p_i t - mq_i, p_j t - mq_j] = -mt[p_i, q_j] - mt[q_i, p_j] = 0 \;.
\end{aligned}$$

These commutators correspond to the Lie algebra of the Galilei group. We notice that in our approach they arise as a consequence of the QAP (through the fundamental commutators between the canonical variables and the Heisenberg equation), and the fact that J, N, P are conserved quantities. This result extends the QAP far beyond the dynamical aspects of the theory, relating it to algebraic aspects too.

6 Conclusion

We presented an equivalent form for the QAP in which the boundary term arising from the variation of the action has the form $F = p_i\delta_0 q_i - \delta_0 p_i q_i$. The absence of the term depending on the Hamiltonian lead us to consider separately the unitary transformation $U = e^{\frac{i}{\hbar}H(t-t_0)}$ as the generator of time translations. One aspect that should be investigated is the case of systems in which the Hamiltonian contains the time explicitly. In some of these cases, quantum mechanics formalism assumes the time evolution is generated by an unitary operator not necessarily having the form $U = e^{-\frac{i}{\hbar}H(t-t_0)}$ but satisfying $U(t,t_0) = 1 - \frac{i}{\hbar}\int_{t_0}^{t} dt'\, H(t')U(t',t_0)$. It remains to be investigated the extension of the QAP to this case.

In our work we considered systems without spin. One approach to describe spin is to add to the canonical variables $\{q_i, p_i\}$ extra fermionic coordinates, $\theta_\alpha, \pi_\alpha$ [5]. The development of a QAP involving both bosonic and fermionic variables is expected to generate supersymmetric quantum mechanics, an area that can provide further applications of the QAP.

Finally, it may be possible to formulate the QAP in quantum field theory following an Hamiltonian formalism which generalizes our approach and whose form can be compared with the lagrangian formalism adopted by Schwinger in [2]. The analysis of the QAP to relativistic quantum fields will be presented in a forthcoming work.

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