systems are also able, by means of their `dissolve` commands, to provide explicit solutions of homogeneous linear constant-coefficient differential equations.

In the classic text *Differential Equations* by Ralph Palmer Agnew (used by the author as a student) the following statement is made:

> It is not reasonable to expect students in this course to have computing skill and equipment necessary for efficient solving of equations such as

\[
4.317 \frac{d^4y}{dx^4} + 2.179 \frac{d^3y}{dx^3} + 1.416 \frac{d^2y}{dx^2} + 1.295 \frac{dy}{dx} + 3.169y = 0. \tag{13}
\]

Although it is debatable whether computing skills have improved in the intervening years, it is a certainty that technology has. If one has access to a computer algebra system, equation (13) could now be considered reasonable. After simplification and some relabeling of output, Mathematica yields the (approximate) general solution

\[
y = c_1 e^{-0.728852x} \cos(0.618605x) + c_2 e^{-0.728852x} \sin(0.618605x)
+ c_3 e^{0.476478x} \cos(0.759081x) + c_4 e^{0.476478x} \sin(0.759081x).
\]

Finally, if we are faced with an initial-value problem consisting of, say, a fourth-order equation, then to fit the general solution of the DE to the four initial conditions, we must solve four linear equations in four unknowns (the \(c_1, c_2, c_3, c_4\) in the general solution). Using a CAS to solve the system can save lots of time. See Problems 59 and 60 in Exercises 4.3 and Problem 35 in Chapter 4 in Review.

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EXERCISES 4.3

In Problems 1–14 find the general solution of the given second-order differential equation.

1. \(4y'' + y' = 0\)
2. \(y'' - 36y = 0\)
3. \(y'' - y' - 6y = 0\)
4. \(y'' - 3y' + 2y = 0\)
5. \(y'' + 8y' + 16y = 0\)
6. \(y'' - 10y' + 25y = 0\)
7. \(12y'' - 5y' - 2y = 0\)
8. \(y'' + 4y' - y = 0\)
9. \(y'' + 9y = 0\)
10. \(3y'' + y = 0\)
11. \(y'' - 4y' + 5y = 0\)
12. \(2y'' + 2y' + y = 0\)
13. \(3y'' + 2y' + y = 0\)
14. \(2y'' - 3y' + 4y = 0\)

In Problems 15–28 find the general solution of the given higher-order differential equation.

15. \(y''' - 4y'' - 5y' = 0\)
16. \(y''' - y = 0\)
17. \(y''' - 5y'' + 3y' + 9y = 0\)
18. \(y''' + 3y'' - 4y' - 12y = 0\)
19. \(\frac{d^2u}{dt^2} + \frac{d^2u}{dt^2} - 2u = 0\)
20. \(\frac{d^3x}{dt^3} - \frac{d^2x}{dt^2} - 4x = 0\)
21. \(y''' + 3y'' + 3y' + y = 0\)
22. \(y''' - 6y'' + 12y' - 8y = 0\)
23. \(y^{(4)} + y''' + y'' = 0\)
24. \(y^{(4)} - 2y''' + y = 0\)
25. \(16 \frac{d^4y}{dx^4} + 24 \frac{d^2y}{dx^2} + 9y = 0\)
26. \(\frac{d^4y}{dx^4} - 7 \frac{d^2y}{dx^2} - 18y = 0\)
27. \(\frac{d^3u}{dt^3} + 5 \frac{d^2u}{dt^2} - 2 \frac{d^2u}{dt^2} - 10 \frac{d^3u}{dt^3} + \frac{du}{dt} + 5u = 0\)
28. \(2 \frac{d^3x}{ds^3} - 7 \frac{d^2x}{ds^2} + 12 \frac{d^3x}{ds^3} + 8 \frac{d^2x}{ds^2} = 0\)

In Problems 29–36 solve the given initial-value problem.

29. \(y'' + 16y = 0, \quad y(0) = 2, \quad y'(0) = -2\)
30. \(\frac{d^2y}{d\theta^2} + y = 0, \quad y\left(\frac{\pi}{3}\right) = 0, \quad y\left(\frac{\pi}{3}\right) = 2\)

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31. \( \frac{d^2y}{dt^2} - 4 \frac{dy}{dt} - 5y = 0, \quad y(1) = 0, y'(1) = 2 \)
32. \( 4y'' - 4y' - 3y = 0, \quad y(0) = 1, y'(0) = 5 \)
33. \( y'' + y' + 2y = 0, \quad y(0) = y'(0) = 0 \)
34. \( y'' - 2y' + y = 0, \quad y(0) = 5, y'(0) = 10 \)
35. \( y''' + 12y'' + 36y' = 0, \quad y(0) = 0, y'(0) = 1, y''(0) = -7 \)
36. \( y''' + 2y'' - 5y' - 6y = 0, \quad y(0) = y'(0) = 0, y''(0) = 1 \)

In Problems 37–40 solve the given boundary-value problem.

37. \( y'' - 10y' + 25y = 0, \quad y(0) = 1, y(1) = 0 \)
38. \( y'' + 4y = 0, \quad y(0) = 0, y(\pi) = 0 \)
39. \( y'' + y = 0, \quad y'(0) = 0, y\left(\frac{\pi}{2}\right) = 0 \)
40. \( y'' - 2y' + 2y = 0, \quad y(0) = 1, y(\pi) = 1 \)

In Problems 41 and 42 solve the given problem first using the form of the general solution given in (10). Solve again, this time using the form given in (11).

41. \( y'' - 3y = 0, \quad y(0) = 1, y'(0) = 5 \)
42. \( y'' - y = 0, \quad y(0) = 1, y'(1) = 0 \)

In Problems 43–48 each figure represents the graph of a particular solution of one of the following differential equations:

(a) \( y'' - 3y' - 4y = 0 \)  
(b) \( y'' + 4y = 0 \)  
(c) \( y'' + 2y' + y = 0 \)  
(d) \( y'' + y = 0 \)  
(e) \( y'' + 2y' + 2y = 0 \)  
(f) \( y'' - 3y' + 2y = 0 \) 

Match a solution curve with one of the differential equations. Explain your reasoning.

43.  
44.  

Discussion Problems

49. The roots of a cubic auxiliary equation are \( m_1 = 4 \) and \( m_2 = m_3 = -5 \). What is the corresponding homogeneous linear differential equation? Discuss: Is your answer unique?

50. Two roots of a cubic auxiliary equation with real coefficients are \( m_1 = -1 \) and \( m_2 = 3 + i \). What is the corresponding homogeneous linear differential equation?
The first $n - 1$ equations in this system, like $y_i u_i' + y_j u_j' = 0$ in (4), are assumptions that are made to simplify the resulting equation after $y_p = u_1(x)y_1(x) + \cdots + u_n(x)y_n(x)$ is substituted in (9). In this case Cramer’s rule gives

$$u_i' = \frac{W_i}{W}, \quad k = 1, 2, \ldots, n,$$

where $W$ is the Wronskian of $y_1, y_2, \ldots, y_n$ and $W_k$ is the determinant obtained by replacing the $k$th column of the Wronskian by the column consisting of the right-hand side of (10)—that is, the column consisting of $(0, 0, \ldots, f(x))$. When $n = 2$, we get (5). When $n = 3$, the particular solution is $y_p = u_1y_1 + u_2y_2 + u_3y_3$, where $y_1, y_2,$ and $y_3$ constitute a linearly independent set of solutions of the associated homogeneous DE and $u_1, u_2, u_3$ are determined from

$$u_1' = \frac{W_1}{W}, \quad u_2' = \frac{W_2}{W}, \quad u_3' = \frac{W_3}{W},$$

(11)

$$W_1 = \begin{vmatrix} 0 & y_2 & y_3 \\ 0 & y_2' & y_3' \\ f(x) & y_2'' & y_3'' \end{vmatrix}, \quad W_2 = \begin{vmatrix} y_1 & 0 & y_3 \\ y_1' & 0 & y_3' \\ f(x) & y_1'' & y_3'' \end{vmatrix}, \quad W_3 = \begin{vmatrix} y_1 & y_2 & 0 \\ y_1' & y_2' & 0 \\ y_1'' & y_2'' & f(x) \end{vmatrix}, \quad \text{and} \quad W = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix}.$$

See Problems 25 and 26 in Exercises 4.6.

**REMARKS**

(i) Variation of parameters has a distinct advantage over the method of undetermined coefficients in that it will always yield a particular solution $y_p$ provided that the associated homogeneous equation can be solved. The present method is not limited to a function $f(x)$ that is a combination of the four types listed on page 141. As we shall see in the next section, variation of parameters, unlike undetermined coefficients, is applicable to linear DEs with variable coefficients.

(ii) In the problems that follow, do not hesitate to simplify the form of $y_p$. Depending on how the antiderivatives of $u_i'$ and $u_i''$ are found, you might not obtain the same $y_p$, as indicated in the answer section. For example, in Problem 3 in Exercises 4.6 both $y_p = \frac{1}{2} \sin x - \frac{1}{4} \sqrt{2} \cos x$ and $y_p = \frac{1}{2} \sin x - \frac{1}{4} \sqrt{2} \cos x$ are valid answers. In either case the general solution $y = y_c + y_p$ simplifies to $y = c_1 \cos x + c_2 \sin x - \frac{1}{4} \sqrt{2} x \cos x.$ Why?

**EXERCISES 4.6**

Answers to selected odd-numbered problems begin on page ANS-5.

In Problems 1-18 solve each differential equation by variation of parameters.

1. $y'' + y = \sec x$
2. $y'' + y = \tan x$
3. $y'' + y = \sin x$
4. $y'' + y = \sec \theta \tan \theta$
5. $y'' + y = \cos^2 x$
6. $y'' + y = \sec^2 x$
7. $y'' - y = \cosh x$
8. $y'' - y = \sinh 2x$
9. $y'' - 4y = \frac{e^{2x}}{x}$
10. $y'' - 9y = \frac{9x}{e^{3x}}$
11. $y'' + 3y' + 2y = \frac{1}{1 + e^t}$
12. $y'' - 2y' + y = \frac{e^x}{1 + x^2}$
13. $y'' + 3y' + 2y = \sin e^x$
14. $y'' - 2y' + y = e^t \arctan t$
15. $y'' + 2y' + y = e^{-t} \ln t$
16. $2y'' + 2y' + y = 4 \sqrt{x}$
17. $3y'' - 6y' + 6y = e^{3x} \sec x$
18. $4y'' - 4y' + y = e^{3/2} \sqrt{1 - x^2}$
In Problems 27 and 28 discuss how the methods of undetermined coefficients and variation of parameters can be combined to solve the given differential equation. Carry out your ideas.

27. $3y'' - 6y' + 30y = 15 \sin x + e^3 \tan 3x$
28. $y'' - 2y' + y = 4x^2 - 3 + x^{-1}e^x$
29. What are the intervals of definition of the general solutions in Problems 1, 7, 9, and 18? Discuss why the interval of definition of the general solution in Problem 24 is not $(0, \infty)$.

30. Find the general solution of $x^4y'' + x^3y' - 4x^2y = 1$ given that $y_1 = x^3$ is a solution of the associated homogeneous equation.

31. Suppose $y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x)$, where $u_1$ and $u_2$ are defined by (5) is a particular solution of (2) on an interval $I$ for which $P$, $Q$, and $f$ are continuous. Show that $y_p$ can be written as

$$y_p(x) = \int_0^x G(x, t)f(t) \, dt,$$  
(12)

where $x$ and $x_0$ are in $I$,

$$G(x, t) = \frac{y_1(t)y_2(x) - y_1(x)y_2(t)}{W(t)},$$  
(13)

and $W(t) = W(y_1(t), y_2(t))$ is the Wronskian. The function $G(x, t)$ in (13) is called the Green’s function for the differential equation (2).

32. Use (13) to construct the Green’s function for the differential equation in Example 3. Express the general solution given in (8) in terms of the particular solution (12).

33. Verify that (12) is a solution of the initial-value problem

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = f(x), \quad y(x_0) = 0, \quad y'(x_0) = 0,$$

on the interval $I$. [Hint: Look up Leibniz’s Rule for differentiation under an integral sign.]

34. Use the results of Problems 31 and 33 and the Green’s function found in Problem 32 to find a solution of the initial-value problem

$$y'' - y = e^{2x}, \quad y(0) = 0, \quad y'(0) = 0$$

using (12). Evaluate the integral.

### 4.7 CAUCHY-EULER EQUATION

**REVIEW MATERIAL**

- Review the concept of the auxiliary equation in Section 4.3.

**INTRODUCTION**

The same relative ease with which we were able to find explicit solutions of higher-order linear differential equations with constant coefficients in the preceding sections does not, in general, carry over to linear equations with variable coefficients. We shall see in Chapter 6 that when a linear DE has variable coefficients, the best that we can usually expect is to find a solution in the form of an infinite series. However, the type of differential equation that we consider in this section is an exception to this rule; it is a linear equation with variable coefficients whose general solution can always be expressed in terms of powers of $x$, sines, cosines, and logarithmic functions. Moreover, its method of solution is quite similar to that for constant-coefficient equations in that an auxiliary equation must be solved.
**EXERCISES 3.3 (PAGE 110)**

1. $x(t) = x_0 e^{-\lambda t}$
   $y(t) = \frac{x_0 \lambda_1}{\lambda_2 - \lambda_1} (e^{-\lambda_1 t} - e^{-\lambda_2 t})$
   $z(t) = x_0 \left(1 - \frac{\lambda_2}{\lambda_2 - \lambda_1} e^{-\lambda_1 t} + \frac{\lambda_1}{\lambda_2 - \lambda_1} e^{-\lambda_2 t}\right)$

3. 5, 20, 147 days. The time when $y(t)$ and $z(t)$ are the same makes sense because most of $A$ and half of $B$ are gone, so half of $C$ should have been formed.

5. $\frac{dx_1}{dt} = 6 - \frac{2}{5} x_1 + \frac{1}{50} x_2$
   $\frac{dx_2}{dt} = \frac{2}{5} x_1 - \frac{2}{5} x_2$

7. (a) $\frac{dx_1}{dt} = 3 \frac{x_2}{100 - t} - 2 \frac{x_1}{100 + t}$
   (b) $x_1(t) + x_2(t) = 150$; $x_2(30) \approx 47.4$ lb

13. $L_1 \frac{di_2}{dt} + (R_1 + R_2)i_2 + R_1i_3 = E(t)$
   $L_2 \frac{di_3}{dt} + R_1i_2 + (R_1 + R_3)i_3 = E(t)$

15. $i(0) = i_0, s(0) = n - i_0, r(0) = 0$

**CHAPTER 3 IN REVIEW (PAGE 113)**

1. $\frac{dP}{dt} = 0.15P$

3. $P(45) = 8.99$ billion

5. $x = 10 \ln \left(\frac{10 + \sqrt{100 - y^2}}{y}\right) - \sqrt{100 - y^2}$

7. (a) $BT_1 + T_2 \frac{BT_1 + T_2}{1 + B}$
   (b) $T(t) = \frac{BT_1 + T_2}{1 + B} + \frac{T_1 - T_2}{1 + B} e^{k(t + \ln r)}$

9. $i(t) = \begin{cases} 4t - \frac{1}{2} t^2, & 0 \leq t < 10 \\ 20, & t \geq 10 \end{cases}$

11. $x(t) = \frac{a e^{\alpha t}}{1 + c e^{\alpha t}}, y(t) = c_2 (1 + c_1 e^{\alpha t})^{y_i/\alpha}$

13. $x = -y + 1 + c_2 e^{-y}$

15. (a) $p(x) = -p(x) g \left(y + \frac{1}{K} \int q(x) \, dx\right)$
   (b) The ratio is increasing; the ratio is constant.
   (d) $\rho(x) = -\frac{K_p}{g(Ky + fq(x)) \, dx}; \quad \rho(x) = \frac{K_p}{2CKp - \beta g}$

**EXERCISES 4.1 (PAGE 128)**

1. $y = \frac{1}{2} e^t - \frac{1}{2} e^{-x}$

3. $y = 3x - 4x \ln x$

9. $(-\infty, 2)$

11. (a) $y = \frac{e^3}{e^2 - 1} (e^t - e^{-t})$  
    (b) $y = \sin x \sin 1$

13. (a) $y = e^x \cos x - e^x \sin x$
   (b) no solution
   (c) $y = e^x \cos x + e^{-\pi/2} e^x \sin x$
   (d) $y = c_2 e^x \sin x$, where $c_2$ is arbitrary

15. dependent  
17. dependent

19. dependent  
21. independent

23. The functions satisfy the DE and are linearly independent on the interval since $W(e^{-3x}, e^{4x}) = 7e^{x} \neq 0$; $y = c_1 e^{-3x} + c_2 e^{4x}$.

25. The functions satisfy the DE and are linearly independent on the interval since $W(e^x \cos 2x, e^x \sin 2x) = 2e^{2x} \neq 0$; $y = c_1 e^x \cos 2x + c_2 e^x \sin 2x$.

27. The functions satisfy the DE and are linearly independent on the interval since $W(x^3, x^5) = x^6 \neq 0$; $y = c_1 x^3 + c_2 x^4$.

29. The functions satisfy the DE and are linearly independent on the interval since $W(x, x^2 - x^2 \ln x) = 9x^6 \neq 0$; $y = c_1 x + c_2 x^2 + c_3 x^3 \ln x$.

35. (b) $y_p = x^2 + 3x + 3e^{2x}$; $y_p = -2x^3 - 6x - \frac{1}{3} e^{2x}$

**EXERCISES 4.2 (PAGE 132)**

1. $y_2 = x e^{2x}$

3. $y_2 = \sin 4x$

5. $y_2 = \sinh x$

7. $y_2 = x e^{2x}$

9. $y_2 = x \cos (\ln x)$

11. $y_2 = 1$

13. $y_2 = x \cos (\ln x)$

15. $y_2 = x^2 + x + 2$

17. $y_2 = e^{2x}, y_p = -\frac{1}{2}$

19. $y_2 = e^{2x}, y_p = \frac{5}{2} e^{3x}$

**EXERCISES 4.3 (PAGE 138)**

1. $y = c_1 + c_2 e^{-x/4}$

3. $y = c_1 e^{3x} + c_2 e^{-2x}$

5. $y = c_1 e^{-ax} + c_2 e^{-4x}$

7. $y = c_1 e^{2x/3} + c_2 e^{-x/4}$

9. $y = c_1 \cos 3x + c_2 \sin 3x$

11. $y = e^{2x} (c_1 \cos x + c_2 \sin x)$

13. $y = e^{-x/2} (c_1 \cos \frac{1}{2} \sqrt{3} x + c_2 \sin \frac{1}{2} \sqrt{3} x)$

15. $y = c_1 + c_2 e^{-x} + c_3 e^{x}$

17. $y = c_1 e^{3x} + c_2 e^{3x} + c_3 x e^{3x}$

19. $u = c_1 e^t + e^{-t} (c_2 \cos t + c_3 \sin t)$

21. $y = c_1 e^{-x} + c_2 x e^{-x} + c_3 x^2 e^{-x}$

23. $y = c_1 + c_2 x + e^{-x/2} (c_3 \cos \frac{1}{2} \sqrt{3} x + c_4 \sin \frac{1}{2} \sqrt{3} x)$

25. $y = c_1 \cos \frac{1}{2} \sqrt{3} x + c_2 \sin \frac{1}{2} \sqrt{3} x$

$+ c_3 x \cos \frac{1}{2} \sqrt{3} x + c_4 x \sin \frac{1}{2} \sqrt{3} x$

27. $u = c_1 e^t + c_2 e^{-t} + c_3 e^{-t} + c_4 e^{2t} + c_5 e^{-5t}$

29. $y = 2 \cos 4x - \frac{1}{2} \sin 4x$

31. $y = -\frac{1}{4} e^{-(t-1)} + \frac{1}{4} e^{3(t-1)}$

33. $y = 0$
35. \( y = \frac{5}{36} - \frac{5}{36} e^{-6x} + \frac{4}{3} xe^{-6x} \)
37. \( y = e^{5x} - x e^{5x} \)
39. \( y = 0 \)
41. \( y = \frac{1}{2} \left( 1 - \frac{5}{\sqrt{3}} \right) e^{-\sqrt{3}x} + \frac{1}{2} \left( 1 + \frac{5}{\sqrt{3}} \right) e^{\sqrt{3}x} \)
   \( y = \cosh \sqrt{3}x + \frac{5}{\sqrt{3}} \sinh \sqrt{3}x \).

**EXERCISES 4.4 (PAGE 148)**

1. \( y = c_1 e^{-x} + c_2 e^{-2x} + 3 \)
2. \( y = c_1 e^{x} + c_2 xe^{x} + \frac{x}{5} + \frac{3}{5} \)
3. \( y = c_1 e^{2x} + c_2 xe^{x} + x^2 - 4x + \frac{7}{4} \)
4. \( y = c_1 \cos \sqrt{3}x + c_2 \sin \sqrt{3}x \) \( + \sqrt{3}x + \left( -4x^2 + 4x - \frac{1}{2} \right) e^{3x} \)
5. \( y = c_1 + c_2 e^{3x} \)
6. \( y = c_1 e^{x} + c_2 xe^{x} + \frac{x}{2} e^{x}x^2 + x^2 \sin x \)
7. \( y = \frac{1}{2} \left( 1 - \frac{5}{\sqrt{3}} \right) e^{-\sqrt{3}x} + \frac{1}{2} \left( 1 + \frac{5}{\sqrt{3}} \right) e^{\sqrt{3}x} \)

**EXERCISES 4.6 (PAGE 161)**

1. \( y = c_1 \cos x + c_2 \sin x + x \sin x + \cos x \ln | \cos x | \)
2. \( y = c_1 \cos x + c_2 \sin x - \frac{1}{2} x \cos x \)
3. \( y = c_1 \cos x + c_2 \sin x + \frac{1}{2} - 6 \cos 2x \)
4. \( y = c_1 e^{x} + c_2 e^{-x} - \frac{x}{5} e^{x}x^2 + x^2 \ln | \cos x | \)

**EXERCISES 4.5 (PAGE 156)**

1. \( (3D - 2)(3D + 2)y = \sin x \)
2. \( (D - 6)(D + 2)y = -6x \)
3. \( D(D + 5)^2 y = e^x \)
4. \( (D - 1)(D - 2)(D + 5)y = \sin x \)
5. \( D(D + 2)(D^2 - 2D + 4)y = 4 \)
6. \( D^4 \)
7. \( D^3(D^2 + 16) \)
8. \( (D + 1)(D - 1)^3 \)
9. \( 1, x, x^2, x^3, x^4 \)
10. \( e^{4x}, e^{-3x/2} \)
11. \( \cos \sqrt{3}x, \sin \sqrt{3}x \)
12. \( 1, e^{5x}, xe^{5x} \)

**EXERCISES 4.7 (PAGE 168)**

1. \( y = c_1 x^{-1} + c_2 x^2 \)
2. \( y = c_1 + c_2 \ln x \)
3. \( y = c_1 \cos(2 \ln x) + c_2 \sin(2 \ln x) \)