Outline

1. Introduction
2. Basic Concepts
3. Mathematical Model
4. Possible Generalizations
5. Calibration
Introduction

- What is this about?
- What is the relationship with Game Theory?
- What are the mathematical tools?
Asset prices (Petrobras, Vale S.A., Itau,...) are naturally random.

How to protect ourselves from large losses?

Buying/Selling Derivatives and Options.

Derivatives and Options are designed to reduce exposure to some source of risk.

How to price options and derivatives appropriately?
“A trading strategy that begins with no money, has zero probability of losing money, and has a positive probability of making money.” Shreve (2004)
Consider a roulette wheel that pays 2 : 1 when the outcome is red and nothing if the outcome is black.

The probabilities of the outcomes are:

\[
\text{Red: 70\% and Black: 30\%}. 
\]

Playing many times, for each $1,00 invested, we expect to receive:

\[
2 \times 0.7 + 0 \times 0.3 = $1,40. 
\]

A gambler sells for $60,00 a ticket that pays $100,00 if red and $0 if black. Is it cheap or expensive?

\[\text{1Souza and Zubelli (2016)}\]
Consider a roulette wheel that pays 2 : 1 when the outcome is red and nothing if the outcome is black. The probabilities of the outcomes are:

Red: 70% and Black: 30%.

Playing many times, for each $1,00 invested, we expect to receive:

\[ 2 \times 0.7 + 0 \times 0.3 = $1.40. \]

A gambler sells for $60.00 a ticket that pays $100.00 if red and $0 if black. Is it cheap or expansive? Using the same ideas as above, the price must be $70.00.

---

\(^1\) Souza and Zubelli (2016)
Arbitrage: an example

Possible situations:

- The gambler keeps the money.
- The gambler bets $60,00 on roulette.
- The gambler bets $50,00 on roulette and keeps $10,00.

---

2 Souza and Zubelli (2016)
Arbitrage: an example²

Possible situations:

- The gambler keeps the money.
- The gambler bets $60,00 on roulette.
- The gambler bets $50,00 on roulette and keeps $10,00.

So, the price is expensive, since it leads to arbitrage.

²Souza and Zubelli (2016)
Arbitrage: an example\textsuperscript{2}

Possible situations:

- The gambler keeps the money.
- The gambler bets $60,00 on roulette.
- The gambler bets $50,00 on roulette and keeps $10,00.

So, the price is expensive, since it leads to \textbf{arbitrage}.

\textbf{What is the fair price?}

\textsuperscript{2}Souza and Zubelli (2016)
Arbitrage: an example

Possible situations:

- The gambler keeps the money.
- The gambler bets $60,00 on roulette.
- The gambler bets $50,00 on roulette and keeps $10,00.

So, the price is expensive, since it leads to arbitrage.

What is the fair price? $50,00. It corresponds to the probabilities 50% – 50%

---

2Souza and Zubelli (2016)
Arbitrage: an example

Possible situations:

- The gambler keeps the money.

- The gambler bets $60,00 on roulette.

- The gambler bets $50,00 on roulette and keeps $10,00.

So, the price is expensive, since it leads to arbitrage.

What is the fair price? $50,00. It corresponds to the probabilities 50\% - 50\%, why?

---

2Souza and Zubelli (2016)
Risk Neutral Probability Measure

- The probability measure that allows to find the correct price of derivatives.
- No arbitrage opportunities or efficient market hypothesis.
- Completeness of Markets
- Uniqueness of prices.
- Hedging: A trading strategy that reduces exposure to the risk of losses.
Vanilla Options

- **European Call**: gives the right, but not the obligation, of buying a share of an asset for a fixed strike price at its maturity.

- **European Put**: similar to the call, but gives the right of selling.

- American Option (call and put) can be exercised any time before its maturity.

- Sometimes, American options are more expensive than the European ones.

- The prices of such contract take into account asset dynamics.

- Other options: Asian, Lookback, Basket, Spread... Real Options...
Typically, the asset price dynamics is given by a semi-martingale:

\[ S_t = \text{something}_t + \text{Martingale}_t. \]

What is a martingale?

\[ \mathbb{E}[|M_t|] < \infty \quad \text{e} \quad \mathbb{E}[M_t | \{ M_l, l \leq s \}] = M_s. \]
Some Hints

Consider a time series of asset prices:

\[ \{ s_{t_0}, s_{t_1}, s_{t_2}, \ldots, s_{t_n} \}, \quad \text{with} \quad \Delta t = t_i - t_{i-1}, \quad i = 1, 2, \ldots, n. \]

Consider now the log-returns:

\[ y_i = \log\left( \frac{s_{t_i}}{s_{t_{i-1}}} \right). \]

Assume that they are:

1. independent,
2. identically distributed
3. Gaussian

So,

\[ \mathbb{E}[y] = \mu \Delta t = \frac{1}{n} \sum_{i=1}^{n} y_i \quad \text{e} \quad \text{Var}[y] = \sigma^2 \Delta t = \frac{1}{n} \sum_{i=1}^{n} y_i^2 - (\mu \Delta t)^2 \]
Since asset price evolves as:

$$\ln(S_t) = \ln(S_0) + \mu t + \text{“randomness”}$$

Such randomness can be modeled as

$$\sigma W_t$$

$W(t)$ is a Brownian motion, i.e.,

1. $W_0 = 0$ a.s.
2. $W_t - W_s$ is $N(0, t - s)$, $t > s$
3. $W_t - W_s$ is independent of $W_s$

Basic tools: SDEs and Itô’s Lemma.
The Itô’s integral can be defined as the “limit” of Riemann’s sums:

\[
\int_0^T X_t dW_t = \lim_{n \to \infty} \sum_{j=1}^n X_{t_j} (W_{t_j} - W_{t_{j-1}})
\]

An SDE is then a stochastic process defined as:

\[
X_T = X_0 + \int_0^T a_t \, dt + \int_0^T b_t \, dW_t
\]

or

\[
dX_t = a_t \, dt + b_t \, dW_t.
\]

The Itô’s Lemma says that, if \( f(t, X) \) is \( C^{1,2} \), then,

\[
df(t, X_t) = \frac{\partial f}{\partial t}(t, X_t) dt + \frac{\partial f}{\partial x}(t, X_t) dX_t + \frac{1}{2} \frac{\partial^2 f}{\partial x^2}(t, X_t) d\langle X_t, X_t \rangle
\]

\[
= \left[ \frac{\partial f}{\partial t}(t, X_t) + \frac{\partial f}{\partial x}(t, X_t) + \frac{1}{2} \frac{\partial^2 f}{\partial x^2}(t, X_t) b_t^2 \right] dt + \frac{\partial f}{\partial x}(t, X_t) b_t \, dW_t
\]
Let \((\Omega, \mathcal{U}, \mathbb{P})\) be a prob. space with filtration \(\mathcal{F} = \{\mathcal{F}_t\}_{t \geq 0}\).

An asset price at time \(t \geq 0\) is given by the SDE:

\[
dS_t = S_t(\mu dt + \sigma dW_t),
\]

where \(W_t\) is a Brownian motion and \(S_0\) is given.

How to price an European option in this setting?
The Black-Scholes Trading Strategy

Main assumptions:

- No transaction costs.
- The asset does not pay dividends.
- No arbitrage.
- Asset price is divisible.
- Short selling is permitted.
- Trading can take place continuously in time.
- Risk-free interest rate and volatility are known and constant.

---

*Korn and Korn (2001)*
Consider the riskless portfolio:

\[ Y_t = \phi_t P_t + \pi_t S_t - C(t, S_t), \]

where \( P_t = P_0 \exp(r \cdot t) \) is a money market account with risk-free interest rate \( r > 0 \)
\( \phi \) and \( \pi \) represent the amount of investment on each asset.

Since the portfolio is riskless

\[ dY_t = rY_t dt. \]

On the other hand,

\[ dY_t = \phi_t dP_t + \pi_t dS_t + dC(t, S_t). \]
The Black-Scholes Trading Strategy

\[ dY_t = \phi_t dP_t + \pi_t dS_t + dC(t, S_t) = \phi_t rP_t dt + \pi_t (\mu S_t dt + \sigma S_t dW_t) \]

\[ - \left[ \frac{\partial C}{\partial t}(t, S_t) dt + \frac{\partial C}{\partial x}(t, S_t) dS_t + \frac{1}{2} \frac{\partial^2 C}{\partial x^2}(t, S_t) \sigma^2 S_t^2 dt \right] \]

\[ = \phi_t rP_t dt + \pi_t (\mu S_t dt + \sigma S_t dW_t) \]

\[ - \left[ \frac{\partial C}{\partial t}(t, S_t) dt + \frac{\partial C}{\partial x}(t, S_t)(\mu S_t dt + \sigma S_t dW_t) + \frac{1}{2} \frac{\partial^2 C}{\partial x^2}(t, S_t) \sigma^2 S_t^2 dt \right] . \]

Making \( \pi_t = \frac{\partial C}{\partial x}(t, S_t) \), and recalling that \( Y_t = \phi_t P_t + \pi_t S_t - C(t, S_t) \), it follows that

\[ 0 = rC(t, S_t) - rS_t \frac{\partial C}{\partial x}(t, S_t) - \frac{\partial C}{\partial t}(t, S_t) - \frac{1}{2} \sigma^2 S_t^2 \frac{\partial^2 C}{\partial x^2}(t, S_t). \]
The Black-Scholes Equation

\[ \frac{\partial C}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + r S \frac{\partial C}{\partial S} - r C = 0, \quad 0 < t < T, \quad S > 0, \]

with terminal condition

\[ C(T, S) = \max\{0, S - K\}. \]

Its solution is given by:

\[ C(t, S) = SN(d_1) - Ke^{-r(T-t)}N(d_2), \]

where,

\[ N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{y^2}{2}} dy, \]

\[ d_1(t, S) = \frac{\log(S/K) + (r + \sigma^2/2)(T-t)}{\sigma \sqrt{T-t}}, \]

and

\[ d_2(t, S) = d_1 - \sigma \sqrt{T-t}. \]
Risk-Neutral Pricing

The no-arbitrage assumption implies the existence of the risk-neutral prob. meas. \( \mathbb{Q} \).

- Under \( \mathbb{Q} \), \( S_t \) satisfies

\[
    dS_t = rS_t dt + \sigma S_t d\tilde{W}_t,
\]

where \( \tilde{W}_t \) is a \( \mathbb{Q} \)-Brownian motion.

- An European call option price is then given by:

\[
    C(t, S_t, T, K) = e^{-r(T-t)} \tilde{E}[\max\{0, S_T - K\} | \mathcal{F}_t].
\]

- More generally, European options with a payoff \( f(S_T) \) satisfy

\[
    V(t, S_t) = e^{-r(T-t)} \tilde{E}[f(S_T) | \mathcal{F}_t].
\]
Feynman-Kac Theorem

Under suitable assumptions, an option with maturity $T$ and payoff $f(S_T)$

$$V(t, S_t) = e^{-r(T-t)}\mathbb{E}[f(S_T) | \mathcal{F}_t].$$

is the solution of the PDE:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + r S \frac{\partial V}{\partial S} - r V = 0, \quad 0 < t < T, \quad S > 0,$$

with terminal condition

$$V(T, S) = f(S).$$
Issues of Black-Scholes Model and Generalizations

- Nice model but too simple.
- Constant Parameters.
- Generalize the model.
- Consider non-constant parameters.
- Possible drawbacks.
- Model calibration.
Let $(\Omega, \mathcal{V}, \mathcal{F}, \tilde{\mathbb{P}})$ be a filtered prob. space.

The asset price $S_t$ satisfies:

$$
\begin{align*}
    dS_t &= (r - q) S_t \, dt + \sigma(t, S_t) S_t \, d\tilde{W}_t, \quad t \geq 0 \\
    S_0 &\text{ is given.}
\end{align*}
$$

Again, European call option price is given by:

$$
C(t, S_t, T, K) = \tilde{\mathbb{E}}[e^{-r(T-t)} \max\{0, S_T - K\} \mid \mathcal{F}_t].
$$

---

Dupire’s Local Volatility Model (1994) 2/2

Fixing $t = 0$ and $S_t = S_0$, it follows that:

$$C(0, S_0, T, K) = e^{-rT} \int_0^\infty \max\{0, S - K\} \varphi(S, T) \, dS$$

and applying Fokker-Planck equation to $\varphi$ and integrating by parts we find:

$$\frac{\partial C}{\partial T} = \frac{1}{2} \sigma^2(T, K; S_0) K^2 \frac{\partial^2 C}{\partial K^2} - (r - q) K \frac{\partial C}{\partial K} - qC, \quad T > 0, \quad K \geq 0$$

$$\lim_{K \to 0} C(T, K) = S_0, \quad T > 0,$$

$$\lim_{K \to +\infty} C(T, K) = 0, \quad T > 0,$$

$$C(T = 0, K) = \max\{0, S_0 - K\}, \quad K > 0.$$
Heston Model (1993)\textsuperscript{6}

- Let \((\Omega, \mathcal{V}, \mathcal{F}, \mathbb{P})\) be a filtered prob. space.

- the asset price \(S_t\) satisfies:

\[
\begin{aligned}
\frac{dS_t}{S_t} &= \mu dt + \sqrt{\nu_t} d\tilde{W}_t, \quad t \geq 0 \\
\frac{d\nu_t}{\nu_t} &= -\lambda(\nu_t - \bar{\nu}) dt + \eta \sqrt{\nu_t} dZ_t,
\end{aligned}
\]

\(S_0, \nu_0\) are given,

\[
\frac{d\langle W_t, Z_t \rangle}{\nu_t} = \rho dt
\]

- Again, European call option price is given by:

\[
C(t, S_t, T, K) = \tilde{\mathbb{E}}[e^{-r(T-t)} \max\{0, S_T - K\} | \mathcal{F}_t].
\]
European options can be priced with PDE’s in some cases.

American options are related to free boundary problems.

For more general models options may not have a PDE representation.

It is necessary to simulate the SDE’s

Option prices can be evaluated through Monte Carlo integration.
- Finite difference and FEM methods to solve PDE problems
- Similar methods can solve free boundary problems.
- SDE’s can be numerically solved by e.g. Euler-Maruyama or Milstein schemes\(^7\).
- Monte Carlo integration.

\(^7\)See Higham (2001)
Model Calibration

1. One of the main problems in Math. Finance and Applied Math. in general.

2. Well-defined direct problem.

3. Calibration problems is in general ill-posed

4. Some regularization technique is needed.
Model Calibration

In Math. Fin. there are two main situations:

- Calibration using historical prices.
  1. Related to risk analysis.
  2. Main techniques: MLE, Kalman Filter, etc.

- Calibration using implicit data (European Call and Put prices).
  1. Related to option pricing.
Let \( \tilde{C} \) be a surface of European call option prices.

Assume that it is given by Dupire’s equation.

So, the corresponding local volatility surface \( \sigma^2_{\text{TRUE}} \), solution of

\[
\tilde{C} = C(\sigma^2_{\text{TRUE}}).
\]

(2)

Unfortunately, only scarce and noisy data \( C^\delta \) is available:

\[
\| \tilde{C} - C^\delta \| \leq \delta,
\]

with \( \delta > 0 \) (noise level).
The inverse problem is ill-posed.

Tikhonov-type regularization leads us to find an element in

$$\arg\min \left\{ \| C(\sigma^2) - C^\delta \|^2 + \alpha f(\sigma^2) : \sigma^2 \in Q \right\},$$

where $Q$ is the set of suitable local vol. surfaces:

$$Q := \{ \sigma^2 \in \sigma_0^2 + H^{1+\epsilon}(D) : a_1 \leq \sigma^2 \leq a_2 \}.$$

Variational theory gives us existence and stability of minimizers, as well as convergence and convergence-rate results.
Numerical Solution

- Dupire’s PDE is solved by a Crank-Nicolson scheme.
- The minimization of the Tikhonov-type functional is solved by the gradient descent method.
- The steps are chosen by Wolfe’s rules.
- The iterations cease whenever the tolerance is satisfied:

\[
\frac{\|C(\sigma^2_k) - C^\delta\|}{\|C^\delta\|} < \text{tol},
\]

typically \(\text{tol} = 0.01\).
Figure: Local vol.: True (left) and reconstructed (right).
Figure: Local vol. reconstruction.
Consider the Heston model:

\[
\begin{align*}
    dS_t &= \mu S_t dt + \sqrt{V_t} S_t dW^1_t, \quad 0 \leq t \leq T_{\text{max}} \\
    dV_t &= \kappa (\theta - V_t) dt + \sigma \sqrt{V_t} dW^2_t,
\end{align*}
\] 

Evaluate the price of \textit{European Asian Options} with strike \(K\), maturity \(T_{\text{max}}\) and payoff

\[
    A(T_{\text{max}}) := \max \left\{ 0, \frac{1}{N} \sum_{j=0}^{N} S_{t_j} - K \right\},
\]

where \(t_j = j \cdot \Delta t\) and \(\Delta t = T_{\text{max}}/N\).
### Pricing Exotic Option

#### Table: Relative errors.

<table>
<thead>
<tr>
<th>log($K/S_0$)</th>
<th>Local Volatility</th>
<th>Black &amp; Scholes</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0</td>
<td>-0.1</td>
</tr>
<tr>
<td>$\tau = 0.1$</td>
<td>0.0247</td>
<td>0.0387</td>
</tr>
<tr>
<td>$\tau = 0.5$</td>
<td>0.0189</td>
<td>0.0317</td>
</tr>
<tr>
<td>$\tau = 1.0$</td>
<td>0.0157</td>
<td>0.0103</td>
</tr>
<tr>
<td>$\tau = 1.5$</td>
<td>0.0400</td>
<td>0.0420</td>
</tr>
</tbody>
</table>


