Local Volatility Calibration in Commodity Markets

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Pricing Problem

Required properties:

- Robustness.
- Reliability.
- Simple calibration.

Desirable property: **implied smile adherence**.

A well-known model in equity markets: *Dupire’s Local Volatility.*

Applications: Calendar spread options, path dependent options, ...
For each future we have options with only one maturity.

- WTI oil: three business days before the termination of trading in the underlying futures contract.
- HH natural gas: the business day immediately preceding the expiration of the underlying futures contract.
- HO heating oil: three business days before the expiration of the underlying futures contract.
- RBOB: three business days before the expiration of the underlying futures contract.

Source: CME webpage.

**Conclusion:** We do not have a surface of option prices on each future.
Important features:

- In commodity markets convenience yield is one important feature.
- Market vanilla option prices are American and then are more expensive than the European ones.
- We need to extract European from American prices.
- The inverse problem associated to American pricing is much harder: There is no framework similar to Dupire’s equation for pricing American options. Then, the forward problem should be solved for each strike and maturity.

We pass to the transformation of American in European prices. This is based on the framework introduced by Black [Bla76].
Black’s Framework

Under the risk-neutral measure, with constant coefficients:

- \( r \) is the risk-free interest rate,
- \( \sigma \) is Black’s volatility,
- \( d \) is the convenience yield.
- \( S_t \) is the commodity spot price, satisfying

\[
dS_t = S_t((r - d)dt + \sigma d\widetilde{W}_t)
\]

- \( F_{t,T} \) is the commodity future, satisfying

\[
dF_{t,T} = \sigma F_{t,T}d\widetilde{W}_t
\]

- They are related by

\[
F_{t,T} = e^{(r-d)(T-t)}S_t
\]

European call options on \( F_{t,T} \) satisfy Black’s equation:

\[
-C_t = \frac{1}{2}\sigma C_{ff}, \quad \text{for } f \geq 0, \ t > 0,
\]

with the terminal condition:

\[
C(T, f) = (f - K)^+, \quad \text{for } f \geq 0.
\]
American options under Black-Scholes [WHD95]:

\[ x := \log(S/K) \text{ and } \tau = (T-t) \frac{1}{2} \sigma^2 \]

Then we have the linear complementary problem:

\[
\begin{aligned}
(u_{\tau} - u_{xx}) &\geq 0, \\
(u(x, \tau) - g(x, \tau)) &\geq 0,
\end{aligned}
\]

\[
\begin{aligned}
(u_{\tau} - u_{xx}) \cdot (u(x, \tau) - g(x, \tau)) &= 0,
\end{aligned}
\]

where, for \( \kappa = (r - d) / (\frac{1}{2} \sigma^2) \),

\[ g(x, \tau) = e^{\frac{1}{4}(\kappa+1)^2 \tau} \left( e^{\frac{1}{2}(\kappa+1)x} - e^{\frac{1}{2}(\kappa-1)x} \right)^+ \text{ for a call.} \]

The boundary conditions are:

\[ u(x, 0) = g(x, 0) \text{ and } \lim_{x \to \pm \infty} u(x, \tau) = \lim_{x \to \pm \infty} g(x, \tau) \]

Then, call prices are given by

\[ C(S, t) = Ke^{-\frac{1}{2}(\kappa-1)x + \frac{1}{4}(\kappa+1)^2 \tau} u(x, \tau) \]
European from American Prices

- Transforming American prices in European ones,
- Then we could use Dupire’s framework.
- Another possibility is the following:
  1. Find the American implied vol. from market option prices.
  2. Then use Black’s formula to find European prices.

\[ C_{\text{AME}} \xrightarrow{\text{B-S AME Pricing}} \sigma_{\text{AME}} \xrightarrow{\text{B-S Formula}} C_{\text{EUR}}. \]
We present also one important feature of Commodity futures.
Correlations

Futures on the same commodity for different maturities are highly correlated.

Figure: Example: Future prices and daily log-returns of Henry Hub nat. gas.
Correlations

Example: Future prices and daily log-returns of Henry Hub nat. gas.

Figure: Minimum of correlations between daily log-returns - first and second tests.
Figure: Example: Future prices and daily log-returns of WTI oil.
Correlations

**Figure:** Minimum of correlations between daily log-returns - first and second tests.

We present now some features of the present model.
Some Features

- Under this framework, the term-structure is given by:
  - The current curve of future prices $F_{0,T}$, for $T > 0$
  - The local volatility surface.

- The model would work fine for short maturity options and a small term-structure curve, since it has only one factor.

- We can form a unique surface of normalized option prices on futures with different maturities.

- Dupire’s formula is not stable in practice, since the inverse problem is ill-posed.

- We apply usual calibration procedures, e.g. Tikhonov regularization.

In what follows, we present the theoretical aspects of the model.
(\(\Omega, \mathcal{V}, \mathcal{F}, \tilde{\mathbb{P}}\)) - risk neutral filtered probability space.
Commodity futures are the underlying assets.
\(F_{t,T}\) denotes the future price at time \(t \geq 0\) with maturity \(T \geq t\).
\(S_t\) denotes the (unknown) spot price at time \(t \geq 0\).
\(F_{t,T} = \tilde{\mathbb{E}}[S_T|\mathcal{F}_t]\), then \(\{F_{t,T}\}_{t \in [0,T]}\) is a martingale.

Then, we assume that, \(F_{t,T}\) satisfies:

\[
\begin{align*}
dF_{t,T} &= \sigma(F_{0,T},t,F_{t,T})F_{t,T}d\tilde{W}_t, \text{ for } 0 \leq t \leq T \\
F_{0,T} \text{ is given and } F_{T,T} &= S_T.
\end{align*}
\]

Now, the PDE for pricing call options.
Fix the current time at $t = 0$, European call options satisfy, with $T \leq T'$:

$$\frac{\partial C}{\partial T} = \frac{1}{2} \sigma^2(F_{0,T'}, T, K) K^2 \frac{\partial^2 C}{\partial K^2}, \quad 0 < T < T', \quad K \geq 0$$

$$\lim_{K \to 0} C(T, K) = F_{0,T'}, \quad 0 < T < T',$$

$$\lim_{K \to +\infty} C(T, K) = 0, \quad 0 < T < T',$$

$$C(T, K) = (F_{0,T'} - K)^+, \quad \text{for } K > 0.$$

We need some technical adaptations.
Perform the change of variables

\[ \tau = T \quad \text{and} \quad y = \log\left(\frac{K}{F_{0,T'}}\right). \]

Then define:

\[ V(F_0,T',\tau,y) := C(F_{0,T'},\tau,F_{0,T'e^y}) \quad \text{and} \quad a(F_0,T',\tau,y) := \frac{1}{2}\sigma^2(F_{0,T'},\tau,F_{0,T'e^y}). \]

Moreover, normalize the option prices by its underlying futures:

\[ V(F_0,T',\tau,y) = V(F_0,T',\tau,y)/F_{0,T'}. \]

Thus, from the previous PDE we have the following problem:
We also assume that

\[ V(F_0, T', \tau, y) = V(S_0, \tau, y) \quad \text{and} \quad a(F_0, T', \tau, y) = a(S_0, \tau, y). \]

Then, \( V \) satisfies:

\[
\begin{align*}
\frac{\partial V}{\partial \tau}(\tau, y) &= a(S_0, \tau, y) \left( \frac{\partial^2 V}{\partial y^2}(\tau, y) - \frac{\partial V}{\partial y}(\tau, y) \right), \quad T > 0, \quad y \in \mathbb{R} \\
\lim_{y \to -\infty} V(\tau, y) &= 1, \quad \tau > 0, \\
\lim_{y \to +\infty} V(\tau, y) &= 0, \quad \tau > 0, \\
V(\tau, y) &= (1 - e^y)^+, \quad \text{for} \ y \in \mathbb{R}.
\end{align*}
\]

It is independent of \( F_0, T' \)!

We present some background properties of the forward operator.
Let $a_1, a_2 \in \mathbb{R}$ be such that $0 < a_1 \leq a_2 < +\infty$.
Consider $a_0$ in $H^{1+\varepsilon}(D)$, with $\varepsilon > 0$ and $a_1 \leq a_0 \leq a_2$.
Define the set
\[ Q := \{ a \in a_0 + H^{1+\varepsilon}(D) : a_1 \leq a \leq a_2 \}. \] (3)

**Proposition ([DCSZ12])**

If $a \in Q$, then Pricing Call Options on futures by Dupire’s Equation is a well-posed problem in $W_{2,\text{loc}}^1(D)$. 

Local Volatility and Commodity Markets

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The Forward Operator (cont.)

Definition

Let $\varepsilon > 0$ and $a_0 \in H^{1+\varepsilon}(D)$ be fixed. Define the forward operator:

$$F : Q \subset H^{1+\varepsilon}(D) \longrightarrow W^{1,2}_2(D)$$

$$a \in Q \quad \rightarrow \quad u(a) - u(a_0) \in L^2(D),$$
The Forward Operator (cont.)

Proposition (From [DCSZ12])
We have the following regularity properties for the forward operator $F$:

(i) It is continuous and compact.

(ii) It is also weakly continuous and weakly closed.

(iii) $F$ is differentiable at $a \in Q$ in every direction $h \in H^{1+\varepsilon}(D)$ such that $a + h \in Q$.

(iv) $F'(a)$ is extensible to a bounded linear operator on $H^{1+\varepsilon}(D)$.

(v) It also satisfies the Lipschitz condition:

$$
\| F'(a) - F'(a + h) \|_{L(H^{1+\varepsilon}(D), L^2(D))} \leq c \| h \|,
$$

for every $h \in H^{1+\varepsilon}(D)$ such that $a + h \in Q$.

Corollary (From [AZ12])
The forward operator $F$ is injective.
The local volatility calibration problem can formulated as follows:

**Problem**

*If \( u \in L^2(D) \) is a surface of European call option prices, then find \( a^\dagger \in Q \), a local volatility surface, satisfying*

\[
    u - u(a_0) = F(a^\dagger),
\]

*with \( a_0 \in Q \) fixed and known.*

Since \( F \) is injective, there exists a unique \( a \in Q \) satisfying Equation (4).
In practice, we observe only the noisy data $u^\delta$, which is related to $u$ by:

$$u^\delta = u + e$$

(5)

$e$ compiles all the uncertainties concerning the measurement of $u^\delta$.

$$\|u - u^\delta\| = \|e\| \leq \delta$$

Problem

Find $a^\dagger \in Q$ satisfying

$$u^\delta - u(a_0) = F(a^\dagger) + e,$$

(6)

with $a_0 \in Q$ fixed and known and $e \in L^2(D)$ unknown with $\|e\| \leq \delta$ and $\delta > 0$. 
Problem

Find one minimizer $a_\alpha^\delta$ in $Q$ for the Tikhonov functional below:

$$F_{a_0,\alpha}^{u_\delta} = \| u(a) - u^\delta \|^2 + \alpha f_{a_0}(a)$$

with $\alpha > 0$ appropriately chosen and $f_{a_0} : \mathcal{D}(f_{a_0}) \subset H^{1+\varepsilon}(D) \to [0, +\infty)$ a suitable functional.
The choice of $\alpha$ is based on the relaxed version of Morozov’s discrepancy principle below:

**Definition**

For $1 < \tau_1 \leq \tau_2$ we choose $\alpha = \alpha(\delta, u^\delta) > 0$ such that

$$\tau_1 \delta \leq \| u(a^\delta_\alpha) - u^\delta \| \leq \tau_2 \delta$$

holds for some $a^\delta_\alpha$ minimizer of the Tikhonov Functional.
The Functional $f_{a_0}$

We assume that:

- $f_{a_0}$ is convex
- $f_{a_0}(a) = 0$ if and only if $a = a_0$.
- It is also coercive, i.e., if $\{a_n\}_{n \in \mathbb{N}}$ satisfy $\|a_n\| \to +\infty$, then $f_{a_0}(a_n) \to +\infty$.
- $f_{a_0}$ is weakly lower semi-continuous, i.e., if $\{a_n\}_{n \in \mathbb{N}}$ converges to $\tilde{a} \in Q$ weakly in $H^{1+\varepsilon}(D)$, then the

  $$f_{a_0}(\tilde{a}) \leq \liminf_{n \to \infty} f_{a_0}(a_n)$$

  holds.
Some Examples

Some canonical examples of $f_{a_0}$ are:

1. **Standard quadratic:**
   
   $$f_{a_0}(a) = \| a - a_0 \|_{L^2(D)}^2.$$

2. **Smoothing quadratic:**
   
   $$f_{a_0}(a) = \beta_1 \| a - a_0 \|_{L^2(D)}^2 + \beta_2 \| \partial_x a - \partial_x a_0 \|_{L^2(D)}^2 + \beta_3 \| \partial_\tau a - \partial_\tau a_0 \|_{L^2(D)}^2.$$

   $\beta_j$ can be arbitrarily chosen and should account discretization levels.

3. **Kullback-Leibler:** denoting $x := (\tau, y) \in D$,
   
   $$f_{a_0}(a) = \int_D \left[ \log(a(x)/a_0(x)) - (a_0(x) - a(x)) \right] dx.$$

4. **Total Variation:**
   
   $$f_{a_0} = \| \partial_x a - \partial_x a_0 \|_{L^1(D)} + \| \partial_\tau a - \partial_\tau a_0 \|_{L^1(D)}.$$
Proposition

The level sets

$$\mu_\alpha(M) = \left\{ a \in Q \mid J_{a_0,\alpha}(a) \leq M \right\}$$

are pre-compact in the weak topology of $H^{1+\epsilon}(D)$. The restriction of $F$ onto $\mu_\alpha(M)$ is weakly continuous.

Theorem (Existence)

Let $\alpha > 0$ and $a_0 \in Q$ be fixed. Then, the Tikhonov functional has a minimizer in $Q$. 
Stability

Definition

A minimizer \( a \in Q \) of the Tikhonov functional is said stable if, for small perturbations on the data \( u \in L^2(D) \), a minimizer of (7) with the perturbed data is in the neighborhood of \( a \).

Theorem (Stability)

Every minimizer of the Tikhonov functional (7) is stable.
Theorem ([AZ12])

The regularizing parameter $\alpha = \alpha(\delta, \mathcal{U}^\delta)$ obtained through Morozov’s discrepancy principle satisfies:

$$\lim_{\delta \to 0^+} \alpha(\delta, \mathcal{U}^\delta) = 0 \quad \text{and} \quad \lim_{\delta \to 0^+} \frac{\delta^2}{\alpha(\delta, \mathcal{U}^\delta)} = 0.$$ 

Theorem ([AZ12])

Let $\{\delta_k\}_{k \in \mathbb{N}}$ be a sequence of positive numbers converging monotonically to 0. Let $\{u^\delta_k\}_{k \in \mathbb{N}}$ be the associated sequence of noisy data. Then, the sequence of minimizers $\{a^\delta_{\alpha_k}\}_{k \in \mathbb{N}}$ converges weakly to $a^\dagger$, the true solution.

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2. V.A. & J.P. Zubelli, Local Volatility Models in Commodity Markets and Online Calibration. Working article
Definition

Let \( \{X_m\}_{m \in \mathbb{N}} \) be a sequence of finite dimensional subspaces of \( H^{1+\varepsilon}(D) \), such that

\[
X_m \subset X_{m+1} \quad \text{for } m \in \mathbb{N} \quad \text{and} \quad \bigcup_{m \in \mathbb{N}} X_m = H^{1+\varepsilon}(D).
\]

Define also the finite-dimensional domains \( Q_m := Q \cap X_m \).

We assume that \( Q_m \neq \emptyset \) for every \( m \in \mathbb{N} \).

Definition ([ACZ13])

Let \( \delta > 0 \), \( u^\delta \) and be fixed. For \( 1 < \tau \leq \lambda \), then choose \( \alpha = \alpha(\delta, u^\delta) > 0 \) and \( m \in \mathbb{N} \) such that

\[
\tau_1 \delta \leq \| F(a_m^\delta, \alpha) - u^\delta \| \leq \lambda \delta,
\]

holds for \( a_m^\delta, \alpha \) a minimizer of the Tikhonov functional in \( Q_m \).
Convergence: The Discrete Case\(^3\) (cont.)

**Theorem**

Let \(\{\delta_k\}_{k \in \mathbb{N}}\) be a sequence of positive numbers converging monotonically to 0. Let \(\{u^\delta_k\}_{k \in \mathbb{N}}\) be the associated sequence of noisy data. Then, if \(m_k\) and \(\alpha_k\) are chosen through the discrepancy principle above, the associated finite-dimensional minimizers \(\{a^\delta_{m_k, \alpha_k}\}_{k \in \mathbb{N}}\) converge weakly to \(a^\dagger\), the true solution.

\(^3\)V.A., A. De Cezaro & J.P. Zubelli, *Discrepancy Based Choice for Domain Discretization Level and Regularization Parameter. Working article.*
Convergence Rates

Theorem (Convergence Rates [AZ12])

Assume that $\alpha = \alpha(\delta, u^\delta)$ is chosen through the Morozov’s discrepancy principle. Furthermore, assume that $f_{A_0}(a) = \|a - a_0\|_{H^{1+\varepsilon}(D)}^2$. Then

$$\|a_\alpha^\delta - a^\dagger\|_{H^{1+\varepsilon}(D)} = O(\delta^{1/2}) \quad \text{and} \quad \|u(a_\alpha^\delta) - u^\delta\| = O(\delta),$$

where $a_\alpha^\delta \in Q$ is the regularized solution.
Under the discrete setting, we have the following result.

**Theorem (Convergence Rates [ACZ13])**

Assume that $\alpha = \alpha(\delta, u^\delta)$ and the discretization level $m = m(\delta, u^\delta)$ are chosen through the discrepancy principle above. Furthermore, assume that $f_{a_0}(a) = \|a - a_0\|_{H^1(\mathcal{D})}^2$ and there exists $a \in Q_m$ such that $\|u(a) - u^\delta\| \leq \varepsilon \delta$ and $f_{a_0}(a) < f_{a_0}(a^\dagger)$, with $1 < \varepsilon < \tau$.

Then

$$\|a_m^\delta, \alpha - a^\dagger\|_{H^1(\mathcal{D})} = O(\delta^{1/2}) \quad \text{and} \quad \|u(a_m^\delta, \alpha) - u^\delta\| = O(\delta),$$

where $a_m^\delta, \alpha \in Q_m$ is the finite-dimensional regularized solution.
Convergence Rates


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Online Approach

How to improve results even further?
Introducing more information:

$$\mathcal{F}_{\mathcal{A}_0,\alpha}(\mathcal{A}) = \int_{S_{\min}}^{S_{\max}} \| u(a(s)) - u^\delta(s) \|^2 ds + \alpha \mathcal{F}_{\mathcal{A}_0}(\mathcal{A}),$$

(10)

In the discrete case:

$$\mathcal{F}_{\mathcal{A}_0,\alpha}(\mathcal{A}) = \sum_{j=1}^{M} \| u(a(s_j)) - u^\delta(s_j) \|^2 + \alpha \mathcal{F}_{\mathcal{A}_0}(\mathcal{A}_M),$$

(11)

with $S_{\min} \leq s_j \leq S_{\max}$ for every $j = 1, \ldots, M$.

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$^2$V.A. & J.P. Zubelli, Local Volatility Models in Commodity Markets and Online Calibration. Working article
Online Approach (cont.)

**Theorem**

There exists a solution for a minimizer for the online Tikhonov functional. It is stable.

Assume that $\delta \to 0$ and $\alpha$ is chosen through the Morozov’s discrepancy principle.

Then the regularized solutions converge weakly to the solution of the noiseless inverse problem $\mathcal{A}^\dagger \in \Omega$.

In addition, when $F_{\mathcal{A}_0}(\mathcal{A}) = \| \mathcal{A} - \mathcal{A}_0 \|_{H'(0,T;H^{1+\epsilon}(D))}^2$, these solutions satisfy the convergence rate:

$$\| \mathcal{A}_\alpha^\delta - \mathcal{A}^\dagger \| = O(\sqrt{\delta}).$$

The same holds in the discrete case.
Synthetic Data: Local Volatility.

Figure: Left: Original. Center and right.: Reconstructions with noisy data.
Figure: Left: Original. Center and right.: Reconstructions with noisy data.
Figure: Left: Residual $\times$ discretization level. Right: Error $\times$ discretization level.
Figure: More data, better results!
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Figure: $L^2$ distance between original and reconstructed local vol.
Figure: Left: Local Volatility. Right: Implied Vol. of model (cont.) and market (squares).
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Figure: Left: Local Volatility. Right: Implied Vol. of model (cont.) and market (squares).
Figure: Left: local vol. reconstructed for some maturities. Right: reconstructed local vol. surface.
Figure: Implied vol. (Black) for market prices (dashed) and model prices (continuous) for two maturities.
Figure: Left: Local Volatility. Right: Implied Vol. of model (cont.) and market (squares).
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HH Local and Implied Volatilities

Figure: Left: Local Volatility. Right: Implied Vol. of model (cont.) and market (squares).
Conclusions

- Dupire’s local vol. applied to commodity markets.
- Implemented American to European prices transformation.
- Local vol. calibration solved by convex regularization.
- Online approach.
- Morozov’s discrepancy principle.
- Numerical tests.


