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ON PRIME AND SEMIPRIME MODULES AND COMODULES

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In this paper we describe the structure of prime and semiprime R-modules M such that $R/Ann_R(M)$ is artinian. The obtained results are then applied to describe the structure of prime and semiprime right comodules over a coring C under some assumption, where a right comodule M is said to be prime (semiprime) if the corresponding left $^*C$-module is prime (semiprime). Finally we apply the results to coalgebras over commutative rings.

Keywords: Prime modules; prime comodules; semiprime modules; semiprime comodules; corings; coalgebras.

0. Introduction

Prime, strongly prime, semiprime and strongly semiprime modules and algebras have been studied extensively in the last few years (see, for example, [3–5], [8] and [14], and the literature quoted in these works).

For coalgebras and comodules over fields, as far as we know, a dual notion of the notion of prime algebras, the coprime coalgebras, was introduced and studied first in [10] and next in [6], but we do not know any paper extending the above notions to corings and comodules.

In this paper we follow a different approach from the ones used in [10] and [6]. We will consider comodules over R-corings C, with some additional assumption, which gives a more general context of that of comodules over coalgebras over fields. Instead of taking a dual of the definition of prime modules to define a coprime comodule, we consider a right comodule M over a coring C as a left $^*C$-module, the dual algebra of C, and say that M is prime (semiprime, etc.) if it is prime (semiprime, etc.) as a module over ${}^*C$.

Section 1 is an introductory section. In Sec. 2 we consider left R-modules M such that $R/Ann_R(M)$ is left artinian, where R is a ring. We describe prime (semiprime) modules of this type in terms of simple modules. We prove that such a module M is prime (semiprime) if and only if it is a direct sum of simple (semisimple) isomorphic R-modules.

In Sec. 3, similar results are obtained for comodules over corings C over a ring R, provided that the coring satisfies the left α-condition and R is a left perfect ring. For example, a comodule is prime (in the above sense) if and only if it is a direct sum of isomorphic simple comodules. Finally, in Sec. 4 we consider coalgebras over commutative rings. The main result shows that, also under some additional assumption, a coalgebra is prime if and only if it is simple and semisimple as a right comodule over itself. The results proved in Secs. 3 and 4 hold, of course, when C is a coalgebra over a field.

1. Prerequisites

For a ring R, modules over R will be considered as left R-modules and R-*Mod* will denote the category of left R-modules. A left R-module M is said to be prime if $Ann_R(M) = Ann_R(N)$ for all nonzero submodule N of M, and is said to be semiprime if $rRrm = 0$, for $r \in R$ and $m \in M$, implies that $rm = 0$. It is easy to see that a semisimple module is semiprime.

Given an R-module M, $\sigma[M]$ denotes the Wisbauer category of $M(*)$, i.e. the category whose objects are all those R-modules subgenerated by M (all R-modules which are isomorphic to submodules of M-generated modules) ([13], p. 118). A module M is said to be strongly prime if M is subgenerated by any of its nonzero submodules, i.e. $M \in \sigma[K]$ for all nonzero submodule K of M ([14], p. 95).

For a module N in $\sigma[M]$ we will denote the M-injective hull of M by $\widehat{N}$, i.e. $\widehat{N}$ is injective in $\sigma[M]$ and an essential extension of N ([13], 17.8).

The following definitions can be found with more details in ([14], 9.1). A class $\mathcal{T}$ of modules in $\sigma[M]$ is called a

(i) pretorsion class if $\mathcal{T}$ is closed under direct sums and factor modules;
(ii) hereditary pretorsion class if $\mathcal{T}$ is closed under direct sums, factor modules and submodules;
(iii) hereditary torsion class if $\mathcal{T}$ is closed under direct sums, factor modules, submodules and extensions in $\sigma[M]$.

The following result gives some equivalent conditions for a module to be strongly prime.

(*)This name has been recently suggested by P. Smith.

Theorem 1.1. ([14], 13.3) *For an R-module M with M-injective hull $\widehat{M}$, the following are equivalent*:

(i) *M is strongly prime*;

(ii) *$\widehat{M}$ is generated by each of its nonzero submodules*;

(iii) *$\widehat{M}$ has no non-trivial fully invariant submodules*;

(iv) *For any pretorsion (hereditary pretorsion) class $\mathcal{T}$ in $\sigma[M]$, $\mathcal{T}(\widehat{M}) = \widehat{M}$ or $\mathcal{T}(\widehat{M}) = 0$ ($\mathcal{T}(M) = M$ or $\mathcal{T}(M) = 0$)*;

(v) *For each $0 \neq x \in M$ and $y \in M$ there exist $r_1, r_2, \ldots, r_n \in R$ such that $\bigcap_{i=1}^{n} Ann_R(r_i x) \subseteq Ann_R(y)$.*

By Theorem 1.1, a left R-module P is strongly prime if and only if for any $x, y \in P$, $x \neq 0$, there exists $a_1, \ldots, a_n \in R$ such that $\bigcap_{j=1}^{n} Ann_R(a_j x) \subseteq Ann_R(y)$. A stronger definition is given in [4]. In fact, in Definition 1.6 of that paper P is said to be strongly prime if for any $0 \neq x \in P$ there exists $a_1, \ldots, a_n \in R$ such that $\bigcap_{j=1}^{n} Ann_R(a_j x) = Ann_R(M)$. In this case we say here that P is strongly prime in the sense of Dauns.

Using (iv) of the above theorem the following is easy to prove.

Corollary 1.2. *Any direct sum of isomorphic strongly prime modules is a strongly prime module.*

Now we recall the definition of strongly semiprime modules. For an R-module M we put $T = End_R(\widehat{M})$, the set of all R-endomorphisms of the M-injective hull $\widehat{M}$ of M. Let K be a submodule of $\widehat{M}$ and $L \in \sigma[M]$. The trace of $\sigma[K]$ in L is defined by

$$\mathcal{T}^K(L) = \sum\{U \subseteq L \mid U \in \sigma[K]\}.$$

The hereditary torsion class in $\sigma[M]$ determined by $\widehat{TK} \subseteq \widehat{M}$ is given by

$$\mathcal{T}_K(L) = \sum\{U \subseteq L \mid Hom_R(U, \widehat{TK}) = 0\}.$$

We have the following:

Proposition 1.3. ([14], 14.3) *The following conditions are equivalent*:

(i) $M/\mathcal{T}_K(M) \in \sigma[K]$.

(ii) $\widehat{M} = \widehat{TK} \oplus \mathcal{T}_K(\widehat{M})$.

A module M is said to be strongly semiprime if the equivalent conditions above are satisfied for any submodule K of M. It is easy to see that semisimple modules are strongly semiprime.

For the basic notions on corings, coalgebras and comodules we refer the reader to [1] and [2]. Here we consider a coring over a left perfect ring R that satisfies an additional condition called the left α-condition. We will quote [1] several times. In so doing, we refer to particular results there by writing simply (19.2) meaning (19.2) of [1], for example (in this case to the definition of the left α-condition).

2. Prime and Semiprime Modules

In this section we denote by $\mathcal{C}_R$ the full subcategory of the category R-*Mod* whose objects are all left R-modules M such that $R/Ann_R(M)$ is a left artinian ring. Of course, if R is left artinian, then $\mathcal{C}_R = R$-*Mod*. As usual the class of objects of $\mathcal{C}_R$ will be denoted simply by $\mathcal{C}_R$.

Recall that an abelian category $\mathcal{A}$ is an additive category such that if $f : X \to Y$ is a morphism in $\mathcal{A}$, then $Ker(f)$ and $Coker(f)$ are in $\mathcal{A}$ and the induced morphism $\bar{f} : Coim(f) \to Im(f)$ is an isomorphism.

Proposition 2.1. *The category $\mathcal{C}_R$ is an abelian category.*

Proof. It is enough to prove that $\mathcal{C}_R$ is closed under subobjects, factor objects and finite direct sums. If $M \in \mathcal{C}_R$ and N is a submodule of M, then we have $Ann_R(N) \supseteq Ann_R(M)$ and $Ann_R(M/N) \supseteq Ann_R(M)$. It follows that $N, M/N \in \mathcal{C}_R$. Also suppose $M_i \in \mathcal{C}_R$, $i = 1, 2$. Then $Ann_R(M_1 \oplus M_2) = Ann_R(M_1) \cap Ann_R(M_2)$ and so $R/Ann_R(M_1 \oplus M_2)$ is left artinian. Thus $M_1 \oplus M_2 \in \mathcal{C}_R$. □

In general $\mathcal{C}_R$ is not closed under arbitrary direct sums. To see this it is enough to take a ring R which is not left artinian but which has an infinite family of left ideals $(I_i)_i$ such that $\bigcap_i I_i = 0$ and R/I_i is left artinian for any i. Then $\sum_i \oplus R/I_i$ is not in $\mathcal{C}_R$.

On the other hand, it is clear that if $(M_i)_{i\in\Omega}$ is any family of isomorphic left R-modules in $\mathcal{C}_R$, then $\sum_{i\in\Omega} \oplus M_i \in \mathcal{C}_R$.

Corollary 2.2. *Assume that $M \in \mathcal{C}_R$. Then $\sigma[M]$ is a subcategory of $\mathcal{C}_R$.*

Proof. By the above remark any direct sum of copies of M is in $\mathcal{C}_R$. Apply Proposition 2.1 to complete the proof. □

Recall that a ring R is said to be right perfect if $R/J(R)$ is left artinian and $J(R)$ is right T-nilpotent, where $J(R)$ denotes the Jacobson radical of R. It is well-known that if R is a right perfect ring, then any nonzero left R-module contains a simple submodule ([8], Theorem 23.20).

Example 2.3. If R is a right perfect ring and M is a prime left R-module, then $M \in \mathcal{C}_R$. In fact, if N is a simple submodule of M we have $Ann_R(M) = Ann_R(N) \supseteq J(R)$ and so $R/Ann_R(M)$ is a factor ring of $R/J(R)$. Hence $R/Ann_R(M)$ is left artinian. We obtain the same conclusion if M is a prime left R-module such that $R/Ann_R(M)$ is a right perfect ring. □

The following gives some properties of modules in $\mathcal{C}_R$.

Proposition 2.4. *Assume that $M \in \mathcal{C}_R$. Then M has a simple submodule. If, in addition, M is a prime module, then $R/Ann_R(M)$ is a simple artinian ring and all simple submodules of M are isomorphic.*

Proof. The first part is clear since under the assumption $R/Ann_R(M)$ is a right perfect ring and so has a simple submodule, and any simple $R/Ann_R(M)$-submodule of M is a simple R-submodule.

Now suppose that M is any prime module in $\mathcal{C}_R$. If N is a simple submodule of M we have $Ann_R(M) = Ann_R(N)$ and so the factor ring $S = R/Ann_R(M)$ is left primitive and artinian, so simple artinian. Also any simple left module over S is isomorphic to N. The result follows. □

Now we can prove one of the main results of this section.

Theorem 2.5. *Assume that $M \in \mathcal{C}_R$. Then the following conditions are equivalent*:

(i) *M is a prime R-module*;
(ii) *M is a strongly prime R-module*;
(iii) *$M = \sum_{i\in\Omega} \oplus M_i$, where $(M_i)_{i\in\Omega}$ is a family of isomorphic simple submodules of M*;
(iv) *M is a sum of isomorphic simple submodules*;
(v) *The ring $R/Ann_R(M)$ is simple.*

Proof. (iii) $\Leftrightarrow$ (iv) and (ii) $\Rightarrow$ (i) are obvious. Proposition 2.4 gives (i) $\Rightarrow$ (v).

(v) $\Rightarrow$ (iii) By assumption $R/Ann_R(M)$ is a simple artinian ring and so any left $R/Ann_R(M)$-module is semisimple. Thus (iii) follows from Proposition 2.4.

(iii) $\Rightarrow$ (ii). This is an easy consequence of Corollary 1.2 and the fact that any simple left R-module is strongly prime. □

Corollary 2.6. *Assume that M is a left R-module such that $R/Ann_R(M)$ is a right perfect ring. Then the conditions* (i)$-$(v) *of Theorem* 2.5 *are equivalent.*

Proof. If M is an R-module satisfying one of the conditions of Theorem 2.5, then M is prime and so by, Example 2.3, $M \in \mathcal{C}_R$. The result follows. □

Any prime module M in $\mathcal{C}_R$ is semisimple. Then M is injective in $\sigma[M]$ and so $M = \widehat{M}$, where $\widehat{M}$ is the injective hull of M in $\sigma[M]$. Thus by Theorem 1.1 we have the following

Corollary 2.7. *Assume that $M \in \mathcal{C}_R$. Then the following conditions are equivalent*:

(i) *M is prime*;
(ii) *M is generated by each of its nonzero submodules*;
(iii) *M has no fully invariant non-trivial submodules*;
(iv) *For any pretorsion class $\mathcal{T}$ in $\sigma[M]$, $\mathcal{T}(M) = M$ or $\mathcal{T}(M) = 0$.*

Now we will consider semiprime modules in $\mathcal{C}_R$. The following is certainly well-known and easy to prove.

Lemma 2.8. *For any R-module M we have*

(i) *If M is semiprime and faithful, then R is a semiprime ring.*
(ii) *M is a semiprime R-module if and only if M is a semiprime module over $R/Ann_R(M)$.*

We denote the socle of M by $s(M)$, i.e. $s(M)$ is the sum of all simple submodules of M. If $M \in \mathcal{C}_R$, by Proposition 2.4, $s(M)$ is an essential submodule of M.

Theorem 2.9. *Assume that $M \in \mathcal{C}_R$. Then the following conditions are equivalent*:

(i) *M is semiprime*;
(ii) *M is semisimple*;
(iii) *M is strongly semiprime.*

Proof. Since (ii) implies (i) and (iii) we only need to prove the converses.

(i) $\Rightarrow$ (ii) If M is semiprime, then by Lemma 2.8 $R/Ann_R(M)$ is a semiprime ring. Also $R/Ann_R(M)$ is left artinian and thus it is semisimple. Hence M is a semisimple R-module.

(iii) $\Rightarrow$ (ii) If M is strongly semiprime, then by ([14], 14.4) M is subgenerated by any essential submodule. Thus M is subgenerated by $s(M)$ and so it is semisimple. □

Corollary 2.10. *Assume that $M \in \mathcal{C}_R$. Then the following conditions are equivalent*:

(i) *M is semiprime*;
(ii) *M is a direct sum of prime submodules*;
(iii) *M is a sum of prime submodules*;
(iv) *Any submodule N of M which is a semiprime module is a direct summand of M.*

Proof. (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) $\Rightarrow$ (iv) are easy to prove since any semiprime module is semisimple.

(iv) $\Rightarrow$ (iii) Suppose that $s(M)$ is a proper submodule of M. Since $s(M)$ is semisimple and so semiprime, by (iv) there exists a submodule N with $s(M) \oplus N = M$. This is a contradiction because $s(M)$ is essential in M. □

A submodule N of a module M is said to be prime in M if M/N is a prime module. In ([3], Proposition 1.20) J. Dauns proved that if the intersection of all submodules which are prime in M is zero, then M is semiprime. Also, in ([4] and [5]) he asked the question of whether the converse is true. The answer to this question is negative as we can see in the example given in ([7], p. 3600). As a consequence of Theorem 2.9 we obtain that the converse holds for modules in $\mathcal{C}_R$.

Corollary 2.11. *Assume that $M \in \mathcal{C}_R$. Then M is semiprime if and only if the intersection of all submodules which are prime in M is zero.*

Proof. If M is semiprime we have $M = \sum_{i\in\Omega} \oplus M_i$, where M_i is simple for any i. Take $N_i = \sum_{j\neq i} \oplus M_j < M$. Then N_i is a submodule which is prime in M and $\bigcap_{i\in\Omega} N_i = 0$. □

The proof of the above corollary also shows that a module in $\mathcal{C}_R$ is semiprime if and only if the intersection of its maximal submodules is zero.

3. Prime and Semiprime Comodules

Throughout this section R denotes an associative algebra (with unit) over a commutative ring and C an R-coring. The set ${}^*C$ of all left R-linear maps from C to R is a ring. Comodules will be always right comodules and they will be considered as left ${}^*C$-modules ([1], Chapter 3).

We will assume in this section that C satisfies the left α-condition which is equivalent to saying that C is locally projective as a left R-module (19.2). In this case C is a flat left R-module and the category of right C-comodules is the full subcategory of the category of left ${}^*C$-modules which is subgenerated by ${}_{^*C}C$ (19.3). Also we will assume that R is a left perfect ring.

We now define different types of comodules using the corresponding definitions for modules.

Definition 3.1. A right C-comodule M is said to be prime (resp. strongly prime, semiprime, strongly semiprime) if the corresponding ${}^*C$-module ${}_{^*C}M$ is prime (resp. strongly prime, semiprime, strongly semiprime).

As we already said in the introduction if k is a field and C is a coalgebra over k, there is a more extensive definition used in the literature (see, for example, the definition of coprime coalgebras given in [6] and [10]). We will see from our results that the definitions used here are much more restrictive. However we will obtain complete results characterizing these types of comodules in terms of simple comodules.

The results of this section are consequences of the results of Sec. 2 because of the following

Proposition 3.2. *Assume that M is a prime right comodule over an R-coring C which satisfies the left α-condition, where R is a left perfect ring. Then ${}^*C/Ann_{^*C}(M)$ is a simple artinian ring.*

Proof. Using (19.12,1) and (19.16,2) we can easily see that the left ${}^*C$-module M has a simple subcomodule (so a simple ${}^*C$-submodule) N and also we can write $N = \sum_{i=1}^{n} x_i R$, for nonzero elements $x_i \in M$, $1 \leq i \leq n$.

Since ${}^*C/Ann_{^*C}(x_i) \simeq {}^*Cx_i = N$ it follows that $Ann_{^*C}(x_i)$ is a maximal left ideal of ${}^*C$, for $1 \leq i \leq n$. Since M is prime $Ann_{^*C}(M) = Ann_{^*C}(N) =$

$\bigcap_{i=1}^{n} Ann_{^*C}(x_i)$ and therefore there is a canonical injection

$$\phi : {}^*C/Ann_{^*C}(M) \to \sum_{i=1}^{n} \oplus {}^*C/Ann_{^*C}(x_i).$$

Consequently ${}^*C/Ann_{^*C}(M)$ is a left artinian ring which is clearly also left primitive. The proof is complete. □

A C-comodule M is called strongly prime in the sense of Dauns if M is a strongly prime ${}^*C$-module in the sense of Dauns.

Now we can prove the main results of this section.

Theorem 3.3. *Assume that C is an R-coring which satisfies the left α-condition and M is a right comodule over C, where R is a left perfect ring. Then the following conditions are equivalent*:

(i) *M is prime*;
(ii) *M is strongly prime*;
(iii) *$M = \sum_{i\in\Omega} \bigoplus M_i$, where $(M_i)_{i\in\Omega}$ is a family of isomorphic simple subcomodules of M*;
(iv) *${}^*C/Ann_{^*C}(M)$ is a simple artinian ring*;
(v) *M is a strongly prime comodule in the sense of Dauns.*

In particular, if the above conditions hold, then M is in the category $\mathcal{C}_{^*C}$.

Proof. It is clear that (iii) ⇒ (ii) ⇒ (i) and (i) ⇒ (iv) by Proposition 3.2. Also (iv) ⇒ (iii) follows from Theorem 2.5. Since (v) clearly implies (i) we need to prove only the converse.

Assume that (i) holds and take $0 \neq x \in M$. Then ${}^*Cx$ is finitely generated as a right R-module and, as in the proof of Proposition 3.2, $Ann_{^*C}(M) = Ann_{^*C}({}^*Cx) = \bigcap_{i=1}^{n} Ann_{^*C}(x_i)$, for some $x_i \in {}^*Cx$. Thus for any i there exist ${}^*c_i \in {}^*C$ such that $x_i = {}^*c_i x$ and (v) follows. The proof is complete. □

Corollary 3.4. *Assume that C is an R-coring which satisfies the left α-condition, where R is a left perfect ring. If there exists a prime right comodule which is faithful as a left ${}^*C$-module, then any left C^*-module is a right comodule.*

Proof. By Theorem 3.3, ${}^*C$ is a simple artinian ring. Hence using (18.12,2) $End_{C^*}(C_{C^*}) = {}^C End(C) \simeq {}^*C$ is simple artinian. Thus by ([13], 20.8), the right C^*-module C is finitely generated. Finally, by the left α-condition, C is locally projective as a left R-module. The result follows from (19.6). □

From Corollary 2.7 we immediately have:

Corollary 3.5. *Assume that M is a right C-comodule, where C and R satisfy the above assumptions. Then the following conditions are equivalent*:

(i) *M is a prime comodule*;
(ii) *M is generated by each nonzero subcomodule of M*;
(iii) *M has no fully invariant non-trivial subcomodules*;
(iv) *For any pretorsion class $\mathcal{T}$ in $\sigma[M]^c$ we have either $\mathcal{T}(M) = M$ or $\mathcal{T}(M) = 0$, where $\sigma[M]^c$ is the subcategory of all the comodules subgenerated by M.*

Assume that C satisfies the left α-condition. If A is a subalgebra of $^*C$ we know that A is dense in $^*C$ (in the finite topology of R^C) if and only if the category of right comodules over C is equal to the category $\sigma[_AC]$ (20.7). On the other hand this category is also equal to $\sigma[_{C^*}C]$ by (19.3).

Corollary 3.6. *Assume that C satisfies the left α-condition, A is subalgebra of $^*C$ which is dense in $^*C$ and R is a left perfect ring. Given a right comodule M over C, the conditions of* (i)$-$(v) *of Theorem* 3.3 *are equivalent for the left A-module M. Moreover, M is a prime comodule if and only if M is a prime A-module.*

Proof. The arguments of Proposition 3.2 can be repeated for the left A-module M. Thus if M is a prime A-module, then $A/Ann_A(M)$ is simple artinian. The proof of the first part can be completed in the same way as in the proof of Theorem 3.3.

On the other hand, since the categories $\sigma[_AC]$ and $\sigma[_{^*C}C]$ are equal the right comodule M is subgenerated by each of its nonzero subcomodules if and only if it is subgenerated by each of its nonzero A-submodules. This shows that M is a strongly prime comodule if and only if it is a strongly prime A-module. □

Now we assume that k is a field and C is a coalgebra over k. For a right comodule M over C put $I = Ann_{C^*}(M)$. Recall that the subcoalgebra

$$Ann_{C^*}(M)^{\perp} = \{c \in C \mid f(c) = 0, \text{ for any } f \in I\}$$

of C is called the coalgebra associated to M. It is well-known that $Ann_{C^*}(M)^{\perp}$ is the smallest subcoalgebra D of C such that M is a right comodule over D.

Let N be a simple comodule. Then the dual algebra $(Ann_{C^*}(N)^{\perp})^*$ is isomorphic to $C^*/Ann_{C^*}(N)$ and so is finite dimensional, and the associated coalgebra $Ann_{C^*}(N)^{\perp}$ is a simple coalgebra. Finally, two simple comodules are isomorphic if and only if they have the same associated coalgebra ([2], Chapter 3).

In this case we have the following:

Corollary 3.7. *Assume C is a coalgebra over a field and M is a right comodule over C. Then the following conditions are equivalent*:

(i) *M is prime*;
(ii) *The coalgebra $Ann_{C^*}(M)^{\perp}$ associated to M is simple*;
(iii) *Any nonzero subcomodule of M has the same associated coalgebra.*

Proof. (i) $\Rightarrow$ (iii) If M is prime and N is any subcomodule of M, then $Ann_{C^*}(N) = Ann_{C^*}(M)$. Thus the coalgebra associated to N is $Ann_{C^*}(M)^{\perp}$ and so (iii) holds.

(iii) $\Rightarrow$ (ii) Let N be a simple subcomodule of M. Then, by (iii), $Ann_{C^*}(M)^{\perp} = Ann_{C^*}(N)^{\perp}$ and also $Ann_{C^*}(N)^{\perp}$ is simple. Hence (ii) follows.

(ii) $\Rightarrow$ (i) Since $Ann_{C^*}(M)^{\perp}$ is simple, $C^*/Ann_{C^*}(M)$ is simple artinian. Also, for any subcomodule N of M, we have $Ann_{C^*}(M) \subseteq Ann_{C^*}(N)$ and so $Ann_{C^*}(M) = Ann_{C^*}(N)$. Hence M is prime. □

Now we come back to comodules over corings and consider semiprime comodules. The next example shows that a semiprime comodule is not necessarily in $\mathcal{C}_{^*C}$.

Example 3.8. Assume that $(C_i)_{i\geq 1}$ is the family of matrix coalgebras $C_i = M^c(i,k)$ over a field k, where $dim_k(C_i) = i^2$, and consider $C = \sum_{i\geq 1} \oplus C_i$, the direct sum of the simple coalgebras C_i. Note that $^*C = C^* \simeq \prod_{i\geq 1} C_i^*$, the direct product of all the algebras C_i^*. It is clear that C is a semiprime left C^*-modules and $Ann_{C^*}(C) = 0$. However $C^*/Ann_{C^*}(C) \simeq C^*$ is not left artinian. □

Even though a semiprime comodule is not necessarily in $\mathcal{C}_{^*C}$, we can reduce to the case of Sec. 2 under an additional assumption: for the rest of the section we assume that C satisfies the left α-condition and R is a right artinian ring.

We can write $M = \sum_{m\in M} {^*C}m$, so that M is semiprime if and only if $^*Cm$ is semiprime, for every $m \in M$. We need the following

Lemma 3.9. *Under the above assumptions, if $^*Cm$ is a semiprime $^*C$-module, then $^*Cm \in \mathcal{C}_{^*C}$.*

Proof. Put $T = {^*C}$. Since Tm is a semiprime module $T/Ann_T(Tm)$ is a semiprime ring. Also, by (19.16,3) Tm has finite length. Hence there exists a finite chain of submodules $0 \subset S_1 \subset S_2 \subset \cdots \subset S_t = Tm$ of Tm with simple factors. As in the proof of Proposition 3.2 $T/Ann_T(S_{i+1}/S_i)$ is left artinian and it follows that $T/Ann_T(Tm)$ is left artinian.

In fact, assume that $t = 2$. Note that $T/Ann_T(S_1)$ and $T/Ann_T(S_2/S_1)$ are left artinian and consider the natural T-homomorphism

$$j : Ann_T(S_1)/Ann_T(S_2) \to T/Ann_T(S_2/S_1)$$

induced by the inclusion. Suppose that $a \in Ann_T(S_1)$ and $a + Ann_T(S_2/S_1) \in T/Ann_T(S_2/S_1)$, then $aTa \subseteq Ann_T(S_2)$. It follows that $a \in Ann_T(S_2)$ since $Ann_T(S_2)$ is semiprime. Hence j is injective and so $T/Ann_T(S_2)$ is left artinian. The result follows by induction. □

Recall that if C satisfies the left α-condition, then a right C-comodule M is semisimple if and only if M is the sum of simple subcomodules (19.13).

Theorem 3.10. *Let M be a right C-comodule, where C satisfies the left α-condition and R is right artinian. Then the following conditions are equivalent*:

(i) *M is semiprime*;
(ii) *M is strongly semiprime*;
(iii) *M is semisimple.*

Proof. (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i) are obvious.

Assume that (i) holds. Then $M = \sum_{m \in M} {}^*Cm$, where ${}^*Cm$ is semiprime for any $m \in M$. By Lemma 3.9 ${}^*Cm$ belongs to $\mathcal{C}_{^*C}$ and so ${}^*Cm$ is a semisimple ${}^*C$-module, by Theorem 2.9. Now the left α-condition for C implies that ${}^*Cm$ is a semisimple comodule and thus M is semisimple. □

Finally, Corollaries 2.10 and 2.11 immediately give the following.

Corollary 3.11. *For a right comodule M the following conditions are equivalent*:

(i) *M is semiprime*;
(ii) *M is a direct sum of prime comodules*;
(iii) *M is a sum of prime comodules*;
(iv) *Any semiprime subcomodule of M is a direct summand.*

Corollary 3.12. *Assume that M is a right C-comodule. Then M is semiprime if and only if the intersection of all subcomodules of M which are prime in M is zero.*

4. Prime and Semiprime Coalgebras

Throughout this section C is a coalgebra over a commutative ring with unit R. It is well-known that C is a right and left comodule over itself. Thus we can consider C as a right and left C^*-bimodule and so a left module over $C^* \otimes C^{*op}$, where C^{*op} denotes the opposite algebra of C^* and $\otimes$ means tensor product over R. We will assume in the following that R is a perfect ring and C satisfies the α-condition.

Definition 4.1. A coalgebra C is said to be prime (semiprime) if it is a prime (semiprime) left module over $C^* \otimes C^{*op}$.

We begin this section with the following

Proposition 4.2. *Assume that C is a coalgebra over a perfect ring R that satisfies the α-condition. Then the following are equivalent*:

(i) *C is prime*;
(ii) *C is a strongly prime left module over $C^* \otimes C^{*op}$*;
(iii) *C is a sum of isomorphic simple left $C^* \otimes C^{*op}$-modules*;
(iv) *$C^* \otimes C^{*op}/Ann_{C^* \otimes C^{*op}}(C)$ is a simple artinian ring*;
(v) *C is generated by any nonzero C^*-subbimodules*;
(vi) *C has no fully invariant non-trivial $C^* \otimes C^{*op}$-submodule*;

(vii) *For any pretorsion class $\mathcal{T}$ in $\sigma[_{C^*\otimes C^{*op}}C]$ we have either $\mathcal{T}(C) = C$ or $\mathcal{T}(C) = 0$;*

(viii) *C is a strongly prime left $C^* \otimes C^{*op}$-module in the sense of Dauns.*

Proof. Assume that C is a prime left $C^*\otimes C^{*op}$-module. Using (4.12) and (41.22,3), we easily see that C has a simple left $C^* \otimes C^{*op}$-submodule N which is finitely generated over R. Now repeating the argument of the proof of Proposition 3.2 we obtain that $C^* \otimes C^{*op}/Ann_{C^*\otimes C^{*op}}(C)$ is a simple artinian ring. The remaining part of the proof follows as in the proof of Theorem 3.3 and Corollary 3.5. □

Condition (ii) of the above proposition should be compared with the definition of strongly prime algebras. In fact, an algebra A over a commutative ring R is said to be strongly prime if A is a strongly prime module over $A \otimes_R A^{op}$ ([14], p. 289).

Now we prove the main result of this section, showing that under our assumption the prime coalgebras are always simple coalgebras.

Theorem 4.3. *Let C be a coalgebra over a perfect ring R that satisfies the α-condition. Then the following are equivalent*:

(i) *C is a simple coalgebra that is right (left) semisimple*;

(ii) *C is a simple module over $C^* \otimes C^{*op}$*;

(iii) *C is a prime coalgebra*;

(iv) *C is prime as right (left) C-comodule.*

Proof. First note that C is faithful as a left (right) C^*-module (4.6,2). Also C is projective over R, since by the α-condition C is flat and R is perfect.

(i) ⇒ (ii) follows from (4.15) and (ii) → (iii) is clear.

(iii) ⇒ (iv) Assume that N is a nonzero left C^*-module of C. If $c^*N = 0$, then $c^*NC^* = 0$, where NC^* is a $C^* \otimes C^{*op}$-submodule of C. Thus $(c^* \otimes 1_{C^*})NC^* = 0$ and so $c^*C = (c^* \otimes 1_{C^*})C = 0$. Therefore $c^* = 0$ and consequently C is a prime left C^*-module.

(iv) ⇒ (i) If C is a prime right comodule, then C is a semisimple right C-comodule by Theorem 3.3. Thus by (4.14) C is a direct sum of simple coalgebras. However C is a prime left C^*-module and so it must be just a simple coalgebra. In fact, if $C = D \oplus E$ for subcoalgebras D and E of C we have that for any $0 \neq d^* \in D^* \subseteq C^*$, $d^*E = 0$ and hence $d^* \in Ann_{C^*}(C)$, which is a contradiction. □

Example 4.4. Consider the matrix coalgebra $C = M^c(n, k)$ with basis $(e_{ij})_{1\le i,j\le n}$, where k is a field. It is not hard to show that C, as a right comodule over itself, is a direct sum of the simple subcomodules I_l, $1 \le l \le n$, where I_l is the k-subspace of C generated by $\{e_{lk} : 1 \le k \le n\}$. □

Assume that C is a coalgebra over a field k. Then we can consider C as a right comodule over the coalgebra $C \otimes_k C^{cop}$, where C^{cop} is the co-opposite coalgebra of C. Thus C is a left module over $(C \otimes_k C^{cop})^*$ and it is known that $C^* \otimes_k C^{*op}$ is a dense subalgebra of $(C \otimes_k C^{cop})^*$ ([2], Ex. 1.3.4) . Then from Corollary 3.6 we immediately have

Corollary 4.5. *Assume that C is a coalgebra over a field k. Then C is a prime coalgebra if and only if C is a prime comodule over $C \otimes_k C^{cop}$.*

Now we consider semiprime coalgebras.

Theorem 4.6. *Assume that C is a coalgebra over a commutative artinian ring R that satisfies the α-condition. Then C is semiprime if and only C is a semisimple left module over $C^* \otimes C^{*op}$.*

Proof. Using (4.12) and (4.16,3) we easily see as in Lemma 3.9 that for any $c \in C$ the factor ring $C^* \otimes C^{*op}/Ann_{C^* \otimes C^{*op}}((C^* \otimes C^{*op})c)$ is left artinian. The proof can be completed as in Theorem 3.10. □

To end the paper we relate our definition of prime coalgebras with the definition of coprime coalgebras used in the literature (see, [10] and [6]).

Let C be a coalgebra over a field k. Recall that for subspaces X and Y of C, the wedge $X \wedge Y$ is defined as

$$X \wedge Y = \Delta^{-1}(X \otimes C + C \otimes Y).$$

The following can be found in [10].

Definition 4.7. A nonzero subcoalgebra P of C is said to be coprime if for any subcoalgebras X and Y of C we have that $P \subseteq X \wedge Y$ implies either $P \subseteq X$ or $P \subseteq Y$.

In particular, C is a coprime coalgebra if for any subcoalgebras X and Y of C with $C = X \wedge Y$ we necessarily have that either $C = X$ or $C = Y$. The notions of prime and coprime coalgebras are related by the following

Proposition 4.8. *Let C be a coalgebra over a field k. Then C is prime if and only if C is coprime and finite dimensional over k.*

Proof. If the coalgebra C is prime, then by Theorem 4.3 C is simple, so finite dimensional, and by ([10], Proposition 1.3) C is coprime. Conversely, if C is a finite dimensional coprime coalgebra, then by Theorem 1.1 of [10] C is simple and so C^* is simple artinian. Hence C is prime. □

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