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Covering Coalgebras and Dual Non-singularity

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Abstract Localisation is an important technique in ring theory and yields the construction of various rings of quotients. Colocalisation in comodule categories has been investigated by some authors (see Jara et al., *Commun. Algebra*, 34(8):2843–2856, 2006 and Nastasescu and Torrecillas, *J. Algebra*, 185:203–220, 1994). Here we look at possible coalgebra covers $\pi : D \to C$ that could play the rôle of a coalgebra colocalisation. Codense covers will dualise dense (or rational) extensions; a maximal codense cover construction for coalgebras with projective covers is proposed. We also look at a dual non-singularity concept for modules which turns out to be the comodule-theoretic property that turns the dual algebra of a coalgebra into a non-singular ring. As a corollary we deduce that hereditary coalgebras and hence path coalgebras are non-singular in the above sense. We also look at coprime coalgebras and Hopf algebras which are non-singular as coalgebras.

Key words localisation of coalgebras · non-singular coalgebras · hereditary coalgebras · path coalgebras · copolyform modules · maximal ring of quotients

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1 Introduction

Embedding algebras into better ones where certain problems have solutions is one of the major tools in ring theory. An analogous tool for coalgebras does not always

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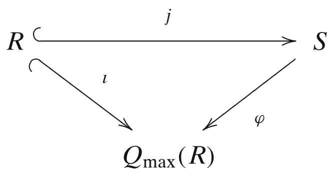
exists. Instead of embedding a coalgebra into a better behaved coalgebra one could also try to find a suitable better behaved coalgebra with a projection onto the first one – a covering coalgebra.

The maximal ring of quotients $Q_{\max}(A)$ of an algebra A is such an example of a universal object that has good properties in particular when the algebra in question is non-singular. Recall that an algebra A is called left non-singular if left annihilators of non-zero elements are never essential as left ideal. This conditions is a kind of non-commutative torsion-freeness for A and Johnson's theorem states that A is left non-singular if and only if $Q_{\max}(A)$ is von Neumann regular, i.e. the weak global dimension of $Q_{\max}(A)$ is zero.

Throughout the text we will assume that rings R are associative and have a unit. Furthermore we shall write homomorphisms of modules opposite of scalars. A submodule N of a left R-module M is called *essential* (*small*) if for all proper non-zero $L \subset M$: $N \cap L \neq 0$ ($N + L \neq M$). We denote a small submodule N of M by $N \ll M$. Given a module M we denote by $\sigma[M]$ the category of submodules of factor modules of direct sums of copies of M (see [20]). For any pair of modules X and Y we denote the trace of X in Y by $\mathrm{Tr}(X, Y) = \sum\{\mathrm{Im}(f) \mid f \in \mathrm{Hom}(X, Y)\}$.

1.1 The Maximal Ring of Quotients

Given a ring R, an overring S of R is called a *left ring of quotients* if $\mathrm{Hom}_{R-}(S/R, S) = 0$. The maximal left ring of quotients $Q_{\max}(R)$ of R is any left ring of quotients such that for any left ring of quotients S of R with embedding $j : R \hookrightarrow S$ there exists a unique ring homomorphism $\varphi : S \to Q_{\max}(R)$ such that $j\varphi = \iota$ where $\iota : R \hookrightarrow Q_{\max}(R)$ denotes the embedding:



The maximal left ring of quotients exists and can be constructed as follows: Let $E = E(R)$ be the injective hull of R as left R-module. Then

$$Q_{\max}(R) := \{x \in E \mid (x)f = 0 \forall f \in \mathrm{End}(E) \textit{ with } (R)f = 0\}.$$

By construction $Q_{\max}(R)$ is the submodule of E that satisfies $Q_{\max}(R)/R = \mathrm{Re}(E/R, E)$. Where $\mathrm{Re}(X, Y) = \bigcap\{\mathrm{Ke}(f) \mid f : X \to Y\}$ denotes the reject of X in Y.

1.2 A Module-theoretic Approach to Covering Coalgebras

A module extension $X \hookrightarrow Y$ is called *dense* if $\mathrm{Hom}(Z/X, Y) = 0$ for all $X \subset Z \subset Y$. In [8] Findlay and Lambek proved that the maximal ring of quotient Q of a ring R is the maximal dense extension of R in the category of R-modules. We will give a module theoretic approach in covering coalgebras using *codense covers* of modules:

A module Y is called a *cover* of X if there exists an epimorphism $\pi : Y \twoheadrightarrow X$. The cover Y is said to be *small* if $\mathrm{Ke}(\pi) \ll Y$ and a cover Y is called a *codense cover* of X if $\mathrm{Ke}(\pi)$ is a *codense submodule* of Y, that is $\mathrm{Hom}(Y, \mathrm{Ke}(\pi)/L) = 0$ for all

$L \subseteq \mathrm{Ke}\,(\pi)$. As a dualisation of dense extensions, codense covers were introduced by Courter in [5] where they are called *co-rational extensions*. Since the term *rational module* has a different meaning in the coalgebraic setting, we prefer to refer to 'dense extensions' and 'codense covers' instead. A non-trivial example of a codense cover is the projection $\mathbb{Q} \twoheadrightarrow \mathbb{Q}/\mathbb{Z}$, which is codense since $\mathrm{Hom}(\mathbb{Q}, \mathbb{Z}/n\mathbb{Z}) = 0$ for all n.

1.3 Some properties of codense covers can be easily checked:

Lemma *Let Z be a cover of X in $\sigma[M]$.*

(1) *If Z is a codense cover of X, then it is a small cover.*
(2) *If Z is a small cover of X and $\pi : X \twoheadrightarrow Y$ is a codense cover then* $\mathrm{Hom}(Z, \mathrm{Ke}\,(\pi)) = 0$.
(3) *If Z is a projective cover of X in $\sigma[M]$ then a cover $\pi : X \twoheadrightarrow Y$ is codense if and only if* $\mathrm{Hom}(Z, \mathrm{Ke}\,(\pi)) = 0$.

Proof

(1) Let $\pi : Z \twoheadrightarrow X$ be a codense cover. Suppose $\mathrm{Ke}\,(\pi) + Y = Z$, then the canonical projection

$$Z \to Z/Y \simeq \mathrm{Ke}\,(\pi)/(\mathrm{Ke}\,(\pi) \cap Y)$$

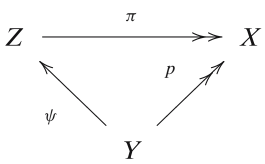
is zero by hypothesis. Thus $Z = Y$ and $\mathrm{Ke}\,(\pi) \ll Z$.
(2) Let $p : Z \to X$ be a small epimorphism and $f \in \mathrm{Hom}(Z, \mathrm{Ke}\,(\pi))$. Extending f to an homomorphism

$$g : X = Z/\mathrm{Ke}\,(p) \to \mathrm{Ke}\,(\pi)/(\mathrm{Ke}\,(p))f,$$

mapping $z + \mathrm{Ke}\,(p) \mapsto (z)f + (\mathrm{Ke}\,(\pi))f$, we have $g = 0$ since X is a codense cover of Y. Thus $(\mathrm{Ke}\,(p))f = \mathrm{Im}\,(f)$. But as $(\mathrm{Ke}\,(p))f \ll \mathrm{Im}\,(f)$, we must have $\mathrm{Im}\,(f) = 0$, i.e. $f = 0$.
(3) Since Z is projective cover of X there exists a small epimorphism $p : Z \twoheadrightarrow X$. For any $U \subseteq \mathrm{Ke}\,(\pi)$ and $f : X \to \mathrm{Ke}\,(\pi)/U$ we have $pf : Z \to \mathrm{Ke}\,(\pi)/U$. Since Z is projective there exists $g : Z \to \mathrm{Ke}\,(\pi)$ which is zero by hypothesis. Hence $pf = 0$ and $f = 0$ as p is an epimorphism. □

1.4 Dual to the definition of a maximal dense extension of a module, we define a maximal codense cover as follows:

Definition 1 Let $X, Y \in \sigma[M]$. A codense cover $p : Y \twoheadrightarrow X$ is called a *maximal codense cover in* $\sigma[M]$ if for any codense cover $\pi : Z \twoheadrightarrow X$ there exists a unique epimorphism $\psi : Y \to Z$ such that $\psi\pi = p$.



Note that our definition differs from Courter's in [5].

1.5 As it was to expect, in case projective covers exist a dual construction like Findlay and Lambek's allows to construct a maximal codense cover for modules:

Theorem *Let $X \in \sigma[M]$ have a projective cover P in $\sigma[M]$. Denote by $\pi : P \to X$ the projection and $T := \mathrm{Tr}(P, \mathrm{Ke}\,(\pi\,))$. Then $\widetilde{P} = P/T$ is a maximal codense cover of X in $\sigma[M]$ with induced epimorphism $\widetilde{\pi} : \widetilde{P} \to X$.*

Proof Note that $\mathrm{Ke}\,(\widetilde{\pi}\,) = \mathrm{Ke}\,(\pi\,)/T$ and as P is a projective cover of $\widetilde{P}$, Hom $(P, \mathrm{Ke}\,(\pi\,)/T) = 0$. By Lemma 1.3 $\widetilde{\pi} : \widetilde{P} \twoheadrightarrow X$ is a codense cover. Let $p : Z \twoheadrightarrow X$ be any other codense cover of X in $\sigma[M]$. By the projectivity of P there exist $\psi : P \to Z$ such that $\psi p = \pi$. As $(T)\psi p = (T)\pi = 0$ we deduce

$$P\,\mathrm{Hom}(P, \mathrm{Ke}\,(\pi\,))\psi = T\psi \subseteq \mathrm{Ke}\,(p\,).$$

Since by Lemma 1.3 $\mathrm{Hom}(P, \mathrm{Ke}\,(p\,)) = 0$, $(T)\psi = 0$. Hence ψ lifts to a homomorphism $\widetilde{\psi} : \widetilde{P} \to Z$ with $\widetilde{\psi} p = \widetilde{\pi}$.

$\widetilde{\psi}$ is unique because for any $\phi : \widetilde{P} \to Z$ with $\phi p = \widetilde{\pi}$ we have $\psi - \phi \in$ Hom $(P, \mathrm{Ke}\,(p\,)) = 0$ by Lemma 1.3 (here we consider ϕ as a map from P to Z). Thus $\psi = \phi$. □

1.6 For a finite dimensional coalgebra C we prove now that $D = \left(Q^r_{max}(C^*)\right)^*$ is a maximal codense cover of C in the category of right C-comodules:

Theorem *Let C be a finite dimensional k-coalgebra, then $D = \left(Q^r_{max}(C^*)\right)^*$ is a finite dimensional coalgebra and there exists a surjective coalgebra homomorphims $\pi : D \to C$ whose kernel is small as a right C-subcomodule of D. In particular $D = \left(Q^r_{max}(C^*)\right)^*$ is a maximal codense cover of C in $\mathcal{M}^C$.*

Proof Since C is finite dimensional, it is a left and right semiperfect coalgebra. Let P be a projective cover of C as right C-comodule with epimorphism $\pi : P \to C$. Since C is finitely generated as left C^*-module, P is also finitely generated as left C^*-module and hence finite dimensional. Since P is a projective right C-comodule, P^* is an injective right C^*-module (by [2, 9.5]). Moreover as $\pi^* : C^* \to P^*$ is an essential embedding, P^* is isomorphic to the injective hull $E(C^*)$ of C^* as right C^*-module. Since $Q^r_{max}(C^*) \subseteq E(C^*)$, it is also finite dimensional. Hence $D = \left(Q^r_{max}(C^*)\right)^*$ is a finite dimensional coalgebra and the transpose $\iota^* : D \twoheadrightarrow C$ of the algebra embedding $\iota : C^* \to Q^r_{max}(C^*)$ is a surjective coalgebra homomorphism. Since ι is an essential monomorphism, π is a small epimorphism.

The kernel K of π is isomorphic to $\left(Q^r_{max}(C^*)/C^*\right)^*$. Note that the dual of any factor comodule $K \twoheadrightarrow L$ is a right C^*-submodule L^* of $Q^r_{max}(C^*)/C^*$. Hence the transpose map of any right C-colinear map $g : C \to K/L$ yields a right C^*-linear map $g^* : (K/L)^* \to C^*$ which could be extended to a right C^*-linear map from $Q^r_{max}(C^*)/C^*$ to $E(C^*)$ and must be zero (where $E(C^*)$ denotes the injective hull of C^* as right C^*-module). Hence D is a codense cover of C. The maximality follows now by a similar argument, taking into account that any codense cover D' of C in $\mathcal{M}^C$ would be finitely generated as comodule and hence finite dimensional. □

1.7 Let K be a field and Γ be a quiver, i.e. a directed graph with finitely many vertices Γ_0 and finitely many arrows Γ_1 and without cycles. The path K-coalgebra

C associated to Γ is the vector space whose basis are all paths in Γ and with comultiplication $\Delta(w) = \sum_{uv=w} u \otimes v$. For each vertex $i \in \Gamma_0$ denote by v_i the unique path of length zero at vertex i. Note that C is finite dimensional and C^* is isomorphic to the path algebra associated to Γ. Since by [11, 13.25] the right maximal ring of quotients of a right artinian right non-singular ring A is isomorphic $\mathrm{End}(\mathrm{Soc}\,(A_A))$, we only need to determine the right socle of C^* to describe the right maximal ring of quotients of C^*. Let A be the path algebra associated to Γ. Denote by Γ_{sink} the set of *terminal* vertices $i \in \Gamma_0$, i.e. those vertices from where no arrow starts. Note that for any $i \in \Gamma_{sink} : v_i A = v_i K$ is a minimal right ideal of A. Moreover for any path p in A which ends at a terminal vertex i, the cyclic right ideal pA is a minimal right ideal and isomorphic to $v_i A$ since both have the same maximal right ideal M_i generated by all paths except v_i. On the other hand let I be a minimal right ideal of A, then $I = \gamma A$ for some linear combination $\gamma = \sum_{j=1}^{n} \lambda_j p_j$ of distinct paths p_j and non-zero coefficients λ_j. Let i' be the vertex where the path p_1 ends and choose a path q from i' to some terminal vertex i. Then $I = \gamma q A$, since I was minimal. Note that $qM_i = 0$ implies that $IM_i = 0$, i.e. the annihilator of I is the maximal right ideal M_i. Hence $I \simeq v_i A$. Moreover γq can be written as a linear combination of paths ending at i, i.e. $\gamma q = \sum \lambda_j p'_j$ where all paths p'_j end at i. Hence $I \subseteq \bigoplus p'_j A$. For any terminal vertex $i \in \Gamma_{sink}$ denote by P_i the set of paths ending at i and set $n_i = |P_i|$. Then we just showed that

$$\mathrm{Soc}\,(A_A) = \bigoplus_{i \in \Gamma_{sink}} \left(\bigoplus_{p \in P_i} pA \right) \simeq \bigoplus_{i \in \Gamma_{sink}} (v_i A)^{n_i}.$$

By [11] the maximal right ring of quotients of A is isomorphic to the endomorphism ring of $\mathrm{Soc}\,(A_A)$:

$$Q^r_{max}(A) \simeq \mathrm{End}(\mathrm{Soc}\,(A_A)) \simeq \prod_{i \in \Gamma_{sink}} \mathrm{End}((v_i A)^{n_i}) \simeq \prod_{i \in \Gamma_{sink}} M_{n_i}(K),$$

where $M_n(K)$ denotes the ring of $n \times n$-matrices over K.

Going back to our path coalgebra we have now a projection of coalgebras of a direct product of matrix coalgebra onto C, i.e.

$$\prod_{i \in \Gamma_{sink}} M^c_{n_i}(K) \twoheadrightarrow C.$$

Here $M^c_n(K) = (M_n(K))^*$ denotes the $n \times n$-matrix coalgebra with basis $\{E_{ij}\}_{1 \le i,j \le n}$, comultiplication

$$\Delta(E_{ij}) = \sum_{l=1}^{n} E_{il} \otimes E_{lj}$$

and counit $\epsilon(E_{ij}) = \delta_{i,j}$.

In case of an infinite dimensional path coalgebra, how can we obtain a covering coalgebra like the matrix coalgebra in our example? For instance for the divided power coalgebra, that is the path coalgebra associated to a single loop. We will see that there is no proper coalgebra cover in the sense defined above.

1.8 We will now turn to some examples of modules that are equal its own maximal codense cover. The next Lemma is probably known, but we were unable to find a reference:

Lemma *Every indecomposable non-faithful injective module over a principal ideal domain is uniserial.*

Proof Let D be a principal ideal domain and M an indecomposable non-faithful injective D-module. By Matlis Theorem [14] $M = E(D/\mathfrak{p})$ for some non-zero prime ideal $\mathfrak{p} = Dp$ of D. Since D is a Dedekind domain, the localisation of D by $\mathfrak{p}$: $D_\mathfrak{p}$ is a discrete valuation ring. Hence $D_\mathfrak{p}$, Q and $Q/D_\mathfrak{p}$ are uniserial $D_\mathfrak{p}$-modules. Take any D-submodule $N \subseteq Q/D_\mathfrak{p}$. We will show that N is also a $D_\mathfrak{p}$-module. For any $a \notin \mathfrak{p} = Dp$ and $n = x/y + D_\mathfrak{p} \in N$ with $y = up^k \in \mathfrak{p}$ and $p \nmid u$. Hence $1 = ra + sp^k$ for some $r, s \in D$. This implies that $\frac{1}{a} - r = \frac{sp^k}{a} \in D_\mathfrak{p}$. Therefore

$$\frac{1}{a}n - rn = \frac{sp^k}{a}\frac{x}{up^k} = \frac{sx}{au} \in D_\mathfrak{p} \Rightarrow \frac{1}{a}n = rn + D_\mathfrak{p}.$$

Hence the action of $1/a$ on an element n in $Q/D_\mathfrak{p}$ is given by a D-scalar multiplication. This shows that $Q/D_\mathfrak{p}$ is a uniserial D-module. Since $Q/D_\mathfrak{p}$ is injective and contains a simple $D_\mathfrak{p}$-submodule which is isomorphic to $D_\mathfrak{p}/\mathfrak{p}D_\mathfrak{p} \simeq D/\mathfrak{p}$, we have that

$$M \simeq E(D/\mathfrak{p}) \simeq Q/D_\mathfrak{p}$$

is a uniserial D-module. Note that all its submodules are of the form $D/\mathfrak{p}^i$. □

1.9 The next theorem states that indecomposable injectives over suitable rings do not have proper codense covers and as we will see below applies in particular to the case of the divided power coalgebra mentioned in 1.7. A module M is called *couniform* or *hollow* if every proper submodule is small.

Theorem *The only possible small covers of a non-faithful indecomposable injective module M over a principal ideal domain D are M and the quotient field Q of D.*

Proof By a theorem of Matlis [14, Prop 3.1] $M = E(D/\mathfrak{p})$ for some maximal ideal $\mathfrak{p}$. Furthermore M is uniserial by 1.8. Let $\pi : P \to M$ be a small cover. Then P is hollow, since M is uniserial and whenever $P = D + E$, $\pi(D) + \pi(E) = M$, i.e. $\pi(D) = M$ or $\pi(E) = M$ and hence $D = P$ or $E = P$ as $\mathrm{Ke}(\pi) \ll P$.

Since M is injective, P is divisible, because for all $0 \neq x \in D$

$$\pi(xP) = x\pi(P) = xM = M,$$

i.e. $xP = P$ as π has a small kernel. As D is a principal ideal domain, P is an indecomposable injective D-module and again by Matlis theorem $P \simeq Q$ or $P \simeq E(D/\mathfrak{q})$ for some maximal ideal $\mathfrak{q}$. In the later case we must have $\mathfrak{p} = \mathfrak{q}$ since

$$D/\mathfrak{p} = \mathrm{Soc}(E(D/\mathfrak{p})) \simeq \mathrm{soc}(E(D/\mathfrak{q})/\mathrm{Ke}(\pi)) = (D/\mathfrak{q}^{i+1})/(D/\mathfrak{q}^i) \simeq D/\mathfrak{q}$$

as $E(D/\mathfrak{q})$ is uniserial and all its submodules are of the form $D/\mathfrak{q}^i$. □

1.10 The *divided power coalgebra* is the path coalgebra C associated to the graph

that is the coalgebra over a field k with basis $\{1, x, x^2, \ldots, x^i, \ldots\}$ and comultiplication:

$$\Delta(x^n) = \sum_{i=0}^{n} x^i \otimes x^{n-i}$$

and counit

$$\epsilon(x^n) = \delta_{0,n}.$$

Corollary *Let C be the divided power coalgebra over a field k. Then C is its own maximal codense cover in the category of C-comodules $\mathcal{M}^C$.*

Proof The dual algebra $C^* \simeq k[[Z]]$ of C is the power series ring in one variable, by the isomorphism:

$$f \mapsto \sum_{n=0}^{\infty} f(x^n) Z^n$$

Note that the power series ring in one variable is a discrete valuation ring, e.g. a principal ideal domain. Since C is an injective cogenerator in $\mathcal{M}^C$ with simple coradical $C_0 = k1$, C is a non-faithful indecomposable injective C^*-module over the discrete valuation ring C^*. By Theorem 1.9 the only small covers of C in C^*-Mod are C and the quotient field Q of C^*. Since C^* is not a C-comodule, Q is also not a C-comodule. Hence the only small cover of C as C-comodule is C itself. □

2 Dual Non-singularity of Modules and Non-singular Coalgebras

Recall that a left R-module M is called *singular* if every element of M is annihilated by an essential left ideal of R. An R-module M is called *non-singular* if it contains no non-zero singular submodule.

2.1 Non-singularity generalises torsion-freeness of modules to the non-commutative setting. Lambek's torsion theory is the right concept for a module theoretic setting in which the construction of maximal dense extension of modules are put. Dual Goldie torsion theories have been studied by various authors [9, 13, 18]. As singular modules play the rôle of torsion modules, small modules will play a similar rôle in the dual situation. Let $\mathcal{S}$ be the class of *small modules* in $\sigma[M]$, i.e. those which are small in their injective hull in $\sigma[M]$. $\mathcal{S}$ is a *Serre class*, i.e. it is closed under submodules, factor modules and extensions (and hence also under finite direct sums). Define

$$\rho(X) = \mathrm{Re}(X, \mathcal{S}) = \bigcap\{U \subseteq X \mid X/U \in \mathcal{S}\}$$

for any $X \in \sigma[M]$ and call X *dual non-M-singular* if $\rho(X) = X$. These are precisely those modules which do not have any non-zero small homomorphic image.

Since an injective module is a direct summand in any extension, injectives are never small. Hence *cohereditary* modules, i.e. those all whose factor modules are injective, are examples of dual non-M-singular modules. On the other hand there exist examples of injective modules that are subdirect products of their M-small factor modules (see Zoeschinger [23]).

2.2 Pushing singularity to smaller categories like $\sigma[M]$ needed a characterisation that was free of referring to left ideals of a ring. Concepts for singularity and their duals had been already proposed in some Abelian categories by Pareigis [17] and it is not difficult to see that in the module case a module M is singular if and only if it is a factor module of a module by an essential submodule. In the case of $\sigma[M]$ it turned out, as shown in [21], that non-singularity of M could be characterised by the internal property that any essential submodule is dense. This property has been studied by Zelmanowitz in [22] where he also termed it *polyform*. It is not difficult to dualise those notions, but it turns out that they are not always equivalent.

2.3 Dual to a polyform module, call a module M *copolyform* if for every small submodule K of M, the canonical projection $M \to M/K$ is a codense cover. Note that dual non-M-singular modules X in $\sigma[M]$ are copolyform since for any small submodule K of X any factor module K/L is also M-small and thus $\mathrm{Hom}(X, K/L) = 0$, i.e. $X \to X/K$ is codense. The converse is not true, e.g. $\mathbb{Z}$ is copolyform, but not non-$\mathbb{Z}$-small. Copolyform modules had been introduced in [12] and were studied also in [19].

2.4 By definition it is clear that copolyform modules can be characterised by their homomorphisms to factor modules. For any two modules X and Y set

$$\nabla(X, Y) = \{f \in \mathrm{Hom}(X, Y) \mid \mathrm{Im}(f) \ll Y\}.$$

This set has been introduced by Beidar and Kasch in [1] were it was termed the *cosingular ideal* of X and Y. Suppose M is copolyform and $f \in \nabla(M, M/N)$ for some $N \ll M$ then $\mathrm{Im}(f) = K/N \ll M/N$ and $N \ll M$ implies $K \ll M$. But as the projection $M \twoheadrightarrow M/K$ is codense, $f \in \mathrm{Hom}(M, K/N) = 0$. Thus $\nabla(M, M/N) = 0$. On the contrary, if $\nabla(M, M/N) = 0$ for all $N \ll M$ then for any small cover $\pi : M \twoheadrightarrow F$ with $K = \mathrm{Ke}\,\pi \ll M$ and submodule $L \subseteq K$ we have $\mathrm{Hom}(M, K/L) \subseteq \nabla(M, M/L) = 0$. Hence $\pi : M \twoheadrightarrow F$ is a codense cover. We have just proved the following statement:

Theorem *An R-module M is copolyform if and only if $\nabla(M, M/N) = 0$ for all $N \ll M$.*

Choosing $N = 0$ in the above Theorem, we get that a copolyform module has no non-zero homomorphism with small image, i.e. $\nabla(M) := \nabla(M, M) = 0$. Note that under some suitable projectivity conditions $\nabla(M)$ equals $\mathrm{Jac}\,(\mathrm{End}(M))$.

2.5 Note that for self-projective modules M, $\nabla(M) = \mathrm{Jac}\,(\mathrm{End}(M))$ (see [20]).

Theorem *A self-projective module M is copolyform if and only if* $\mathrm{Jac}\,(\mathrm{End}(M)) = 0$.

Thus a ring R is copolyform as left R-module if and only if it is semiprimitive.

2.6 Since our aim is to apply the module theoretic terms above to the situation of coalgebras, recall that any coalgebra C of a field k is an injective cogenerator in the category $\mathcal{M}^C$ of right C-comodules. Moreover there exists an anti-isomorphism of rings between the dual algebra C^* and the endomorphism of C as right C-comodule and an isomorphism of rings between C^* and the endomorphism of C as left C-comodule:

$$\mathrm{End}({}_{C^*}C)^{op} \simeq C^* \simeq \mathrm{End}(C_{C^*}).$$

Under some light injectivity and cogenerator properties we can say much more about copolyform modules. A module Q is called *pseudo-injective* with respect to a non-zero monomorphism $f : Y \hookrightarrow X$ if for all non-zero $g : Y \to Q$ there exist $h \in \mathrm{End}(Q)$ and $k \in \mathrm{Hom}(X, Q)$ such that $fk = gh \neq 0$. A module Q is called *pseudo-injective* in $\sigma[M]$ if it is pseudo-injective with respect to all non-zero monomorphism $f : Y \hookrightarrow X$ in $\sigma[M]$.

Lemma *Let M be pseudo-injective in $\sigma[M]$. Then* $\mathrm{Hom}(M/N, M) = 0$ *for all submodules N such that M/N is M-small provided* $\nabla(M) = 0$.

Proof Assume that M/N is small in some module $X \in \sigma[M]$ and let $f : M/N \to M$ be a homomorphism. Suppose f is non-zero then by pseudo-injectivity there are homomorphisms $h \in \mathrm{End}(M)$ and $k \in \mathrm{Hom}(X, M)$ such that $fh = ik \neq 0$ where $i : M/N \hookrightarrow X$ denotes the inclusion. Since homomorphic images of small modules are small, $\mathrm{Im}\,(fh) = \mathrm{Im}\,(ik) \ll M$. Considering the projection $p : M \twoheadrightarrow M/N$ we get a homomorphism $pfh \in \mathrm{End}(M)$ whose image is small in M. Since $\nabla(M) = 0$, $pfh = 0$ which implies $fh = 0$, a contradiction. Thus $\mathrm{Hom}(M/N, M) = 0$. □

2.7 Lemma 2.6 shows that a pseudo-injective module M with $\mathrm{Hom}(M/N, M) \neq 0$ for all non-zero $N \subseteq M$, is dual non-M-singular if and only if $\nabla(M) = 0$. We will show that this is also equivalent to $\mathrm{End}(M)$ being non-singular. Say that a module M is *coretractable* if for all non-zero submodules N of M: $\mathrm{Hom}(M/N, M) \neq 0$. We first need the following Lemma

Lemma *Let M and Q be left R-modules and* $T := \mathrm{End}(Q)$. *Denote by $Z(M^*)$ the singular submodule of* $M^* := \mathrm{Hom}(M, Q)$ *as right T-module. Suppose that Q is coretractable then*

$$Z(M^*) \subseteq \nabla(M, Q)$$

holds. If moreover Q is pseudo-injective with respect to all monomorphisms of the form $g : Q/\mathrm{Ke}\, g \hookrightarrow Q$ for any $0 \neq g \in T$ then equality hold, i.e. $Z(M^*) = \nabla(M, Q)$.

Proof Take $f \in Z(M^*)$. Then $Ann_T(f) = \{g \in T \mid fg = 0\}$ is essential in T. Suppose $\mathrm{Im}\,(f) + U = Q$ for some submodule U of Q. Then $Ann_T(f) \cap Ann_T(U) = Ann_T(\mathrm{Im}\,(f) + U) = 0$ implies $\mathrm{Hom}(Q/U, Q) = Ann_T(U) = 0$. By hypothesis $U = Q$, i.e. $\mathrm{Im}\,(f) \ll Q$ and $f \in \nabla(M, Q)$.

Now assume that Q is pseudo-injective with respect to all monomorphisms $g : Q/\mathrm{Ke}\,(g) \hookrightarrow Q$. Let $f \in \nabla(M, Q)$ and $g \in T$ such that $gT \cap Ann_T(f) = 0$. Suppose there exists a non-zero $h \in Ann_T(\mathrm{Ke}\, g) \cap Ann_T(\mathrm{Im}\,(f))$. As h defines a non-zero homomorphism from $Q/\mathrm{Ke}\, g$ to Q we have by hypothesis endomorphisms $k, l \in T$ such that $0 \neq gk = hl$. But as $h \in Ann_T(\mathrm{Ke}\, g) \cap Ann_T(f)$, we have $hl = gk \in gT \cap Ann_T(f) = 0$; a contradiction. Thus $Ann_T(\mathrm{Ke}\, g) \cap Ann_T(\mathrm{Im}\,(f)) = 0$ and

$$\begin{aligned} 0 &= Ann_T(\mathrm{Ke}\, g) \cap Ann_T(\mathrm{Im}\,(f)) \\ &= Ann_T(\mathrm{Ke}\, g + \mathrm{Im}\,(f)) \\ &\simeq \mathrm{Hom}(Q/(\mathrm{Ke}\, g + \mathrm{Im}\,(f)), Q). \end{aligned}$$

Since Q is coretractable, $\mathrm{Ke}\, g + \mathrm{Im}\,(f) = Q$, but as $\mathrm{Im}\,(f) \ll Q$, $g = 0$. □

Note that the condition in Lemma 2.7 is fulfilled if Q is semi-injective, i.e. injective with respect to all monomorphisms of the above form, or if Q is pseudo-injective in $\sigma[Q]$.

2.8 The last Lemma 2.7 together with 2.6 enables us to characterise those copolyform modules which are injective cogenerators:

Theorem *Let M be a coretractable left R-module that is pseudo-injective in $\sigma[M]$. Then the following statements are equivalent:*

(a) *M is dual non-singular in $\sigma[M]$.*
(b) *M is copolyform.*
(c) $\nabla(M) = 0$.
(d) $\mathrm{End}(M)$ *is a right non-singular ring.*

2.9 Having in mind the fact $\mathrm{End}({}_{C^*}C)^{op} \simeq C^* \simeq \mathrm{End}(C_{C^*})$ we deduce from the last Theorem 2.8 the following

Theorem *Let C be a coalgebra over a field k. Then the following statements are equivalent:*

(a) *C is a copolyform right C-comodule.*
(b) *C^* is a left non-singular ring.*
(c) *C is a copolyform left C-comodule.*

Any coalgebra that satisfies one of the above conditions is called non-singular.

2.10 In [16], Nastasescu et al. called a coalgebra C *hereditary* if C is a cohereditary left (and/or right) C-comodule. By our remark in 2.1 cohereditary modules are dual non-singular. Hence by 2.8 any hereditary coalgebra is non-singular. Chin showed in [3] that any path coalgbera is hereditary. Furthermore Chin and Montgomery showed in [4] that any coalgebra over an algebraically closed field is Morita–Takeuchi equivalent to a subcoalgebra of a path coalgebra. Thus hereditary and hence non-singular coalgebras are ubiquitous. In [16] it has been also proven that a finite dimensional coalgebra C is hereditary if and only if C^* is left hereditary. Since there are finite dimensional algebras which are left non-singular, but not left hereditary, we easily can construct coalgebras which are non-singular but not hereditary.

Call a coalgebra C *cosemiprime* if $I \wedge I \neq C$ holds for all proper subcoalgebras I of C. It is not difficult to see that C is a cosemiprime coalgebra if and only if C^* is semiprime and we deduce that a cocommutative coalgebra is non-singular if and only if C is cosemiprime.

2.11 The strict hierarchies of coalgebraic properties

$$\textit{cosemisimple} \Rightarrow \textit{hereditary} \Rightarrow \textit{non-singular}$$

collapses when assuming some flatness condition on the coalgebra: Since a coalgebra C is flat as right C^*-module if and only if C^* is left self-injective (see [2]), we have that the dual algebra C^* of a non-singular coalgebra C which is flat as left C^*-module must be a left self-injective and left non-singular ring and hence von Neumann regular (as it equals its own maximal left ring of quotient). Note that a von Neumann regular ring is semiprimitive, hence $\text{Jac}\,(C^*) = 0$. By [2], $\text{Jac}\,(C^*) = C_0^\perp$ where C_0 denotes the coradical of C. Hence $C_0^\perp = 0$ implies $C = C_0$. We just proved the following theorem:

Theorem *A coalgebra C is cosemisimple if and only if C is non-singular and flat as right C^*-module.*

Since finite dimensional Hopf algebras are projective as comodule, we deduce that finite dimensional Hopf algebra which are right non-singular coalgebras are cosemisimple.

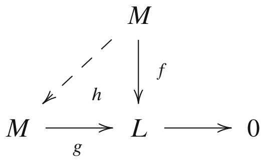
2.12 Note that any coalgebra C can be written as a sum of indecomposable injective comodules E_λ. If C is cocommutative then each of the E_λ is actually a subcoalgebra of C. Assume now that C is a cocommutative semiperfect coalgebra over a field k, then $C = \bigoplus_\lambda E_\lambda$ is a direct coproduct of finite dimensional cocommutative indecomposable coalgebras. If moreover C is non-singular, then each of the E_λ is also non-singular and E_λ^* is a finite dimensional commutative semiprime k-algebra. Thus E_λ^* is a finite field extension K_λ of k and $E_\lambda = K_\lambda^*$ is a finite dimensional simple coalgebra. Thus we have proved the following

Theorem *Any cocommutative non-singular and semiperfect coalgebra is cosemisimple.*

2.13 The lattice of submodules of a module is pseudo-complemented, but its dual lattice does not need to be. To overcome this problem while dualising module theoretic notions, one has to make suitable assumption on the lattice of submodules. An R-module M is called *weakly supplemented* if any submodule N of M has a *weak* *supplement*, that is a submodule L of M such that $N + L = M$ and $N \cap L \ll M$. This is a weak form of a pseudo-complement in the dual lattice of submodules of M.

Theorem *The following statements are equivalent for a weakly supplemented module:*

(a) *M is copolyform.*
(b) $\nabla(M, M/N) = 0$ *for all $N \subseteq M$.*
(c) *Every factor module of M is copolyform.*
(d) $\nabla(M) = 0$ *and M is M-im-small-projective, i.e. any diagram*



with $\mathrm{Im}\,(f) \ll L$ *can be commutatively extended by some* $h : M \to M$.

Proof

(a) $\Rightarrow$ (b) Let $f : M \to M/N$ have small image and choose a weak supplement L of N. Thus $M/N = (N + L)/N \simeq L/(L \cap N) \subseteq M/(L \cap N)$. Since $L \cap N \ll M$ and $f \in \nabla(M, M/(L \cap N)$, we have $f = 0$ by (a).
(b) $\Rightarrow$ (c) let $N \subseteq L \subseteq M$ such that $L/N \ll M/N$ and $f \in \nabla(M/N, M/L)$. Then $f\pi_N \in \nabla(M, M/L) = 0$, i.e. $f = 0$. Hence M/N is copolyform.
(c) $\Rightarrow$ (a) is trivial and $(b) \Rightarrow (d)$ is clear, since for $N = 0$, $\nabla(M, M) = \nabla(M) = 0$ and as $\nabla(M, M/N) = 0$ for all factor modules L of M, there are no non-zero homomorphisms $f : M \to L$ with small image, i.e. M is trivially M-im-small projective.
(d) $\Rightarrow$ (a) Let $f \in \nabla(M, M/N)$ with $N \ll M$ and denote by $\pi_N : M \to M/N$ the canonical projection. By M-im-projectivity there exists $h : M \to M$ such that $\pi_N h = f$. Since $\mathrm{Im}\,(f) = \mathrm{Im}\,(\pi_N h) \ll M/N$ and $N \ll M$, we have $\mathrm{Im}\,(h) \ll M$, i.e. $h \in \nabla(M) = 0$. Thus $f = 0$. □

A module which satisfies condition (c) is also called *strongly copolyform*. This is in general a stronger condition then copolyformness. In [19] strongly copolyform modules are called copolyform.

2.14 A module M is called *couniform* or *hollow* if $N + L = M$ implies $N = M$ or $L = M$ for all proper submodules N, L of M. Uniserial modules are couniform and couniform modules are indecomposable. Furthermore couniform modules are trivially weakly supplemented since all proper submodules are small. From the last characterisation of copolyform modules we easily deduce that a couniform module is copolyform if and only if every projection $M \twoheadrightarrow M/N$ for any proper submodule N of M is codense. Couniform copolyform modules are called *epiform* and satisfy the property that all of their non-zero endomorphisms are epimorphisms. The converse holds under some suitable assumptions as we will see later.

2.15 In case of couniform modules we deduce from 2.8 the following

Corollary *Let M be a couniform coretractable left R-module that is pseudo-injective in $\sigma[M]$. Then the following statements are equivalent:*

(a) *M is dual non-M-singular.*
(b) *M is epiform.*
(c) *Every non-zero endomorphism of M is an epimorphism.*
(d) *Every non-zero homomorphism from a factor module L of M to M is surjective.*
(e) $\mathrm{End}(M)$ *is a domain.*

Proof

(a) ⇔ (b) Follows from 2.8.
(b) ⇒ (c) For any $0 \neq f \in \mathrm{End}(M)$, $\mathrm{Im}(f) \not\ll M$ as $\nabla(M)=0$. Thus $\mathrm{Im}(f)=M$.
(c) ⇒ (d) Let $f: M/N \to M$ since $\pi_N f$ is an epimorphism of M, f has to be an epimorphism (here π_N denotes the projection).
(d) ⇒ (e) If $fg=0$, then $\mathrm{Im}(f) \subseteq \mathrm{Ke}(g)$. And if $f \neq 0$, then $M = \mathrm{Im}(f) = \mathrm{Ke}(g)$, i.e. $g=0$.
(e) ⇒ (a) Follows from 2.8 as domains are non-singular. □

2.16 The characterisation 2.15 of epiform modules yields that a coalgebra C is epiform as right (or left) comodule if and only if C^* is a domain. Recall that a coalgebra C is called *coprime* if C^* is a prime ring. As we see, any coalgebra that is epiform as coalgebra is a coprime coalgebra. In case C is cocommutative those notions are equivalent. We are going to show that there exists a dichotomie for coprime coalgebras that states that over a coprime coalgebra either every comodule is projective or no non-trivial comodule is projective.

Theorem *The following statements are equivalent for a coprime coalgebra C over a field k:*

(a) *C is cosemisimple;*
(b) *C^* is a simple ring.*
(c) *C is finite dimensional.*
(d) *There exists a non-zero projective right (left) C-comodule.*
(e) *There exists a non-zero right (left) C-comodule that is not singular as a C^*-module.*

In this case C^* is a matrix algebra over k.

Proof

(a) ⇒ (b) Since C^* is prime, it is indecomposable as an algebra. Hence C is an indecomposable coalgebra, i.e. C cannot be written as the sum of two non-zero subcoalgebras. By hypothesis, C is a direct sum of simple subcoalgebras and hence must be a simple coalgebra. Thus C^* is a matrix algebra over k.
(b) ⇒ (c) Assume C^* is simple, then C is a simple coalgebra, because if D is any subcoalgebra of C, then $D^\perp$ is an ideal of C^* and hence 0 or C^*, i.e. $D = C$

or $D = 0$. Since any non-zero element of C is contained in a non-zero finite dimensional subcoalgebra of C, C must be finite dimensional.

(c) ⇒ (a) Since C is finite dimensional, C^* is finite dimensional. As C^* is also a prime ring, it must be a matrix algebra.

(a) ⇒ (d) Is clear since every right (left) C-comodule is projective.

(d) ⇒ (e) Is trivial since projective modules are not singular.

(e) ⇒ (c) Suppose M is a non-zero left C-comodule which is not singular as C^*-module. Then there exists a cyclic (and hence finite dimensional) C^*-submodule N of M which is not singular. As the annihilator $\mathrm{Ann}_{C^*}(N)$ is not an essential left ideal of C^*, but all non-zero ideals of a prime ring are essential left ideals, we conclude that $\mathrm{Ann}_{C^*}(N) = 0$, thus

$$C^* = C^*/\mathrm{Ann}_{C^*}(N) \hookrightarrow \oplus_{i=1}^{s} C^*/Ann_{C^*}(n_i)$$

is finite dimensional, where n_i is a generating set of N. □

2.17 By negating (*c*), (*d*), (*e*) of the last Theorem we deduce that a coprime coalgebra C over a field k has infinite dimension if and only if every right or left C-comodule is singular as C^*-module if and only if the category of right or left C-comodules does not contain any non-zero projective object. This shows the dichotomie of coprime coalgebras: Either every comodule is projective and the coalgebra is necessarily a matrix coalgebra or every comodule is singular as C^*-module and $\mathcal{M}^C$ has no non-zero projective object. This dichotomie shows also that we can not use projective cover to build maximal codense covers of infinite dimensional coprime coalgebras. From 2.15 have that any C which is epiform is either the dual of a finite dimensional division algebra K over k or infinite dimensional such that the category of right C-comodules consists of torsion C^*-modules, in particular there are no non-zero projective objects in $\mathcal{M}^C$.

2.18 Copolyform module with projective covers can be characterised by their endomorphism rings.

Proposition *Let M be an R-module with projective cover P in $\sigma[M]$. Then M is copolyform if and only if* $\mathrm{Jac}\,(\mathrm{End}(P)) = 0$.

Proof Recall that $\mathrm{Jac}\,(\mathrm{End}(P)) = \nabla(P)$. Assume M to be copolyform and let $f \in \nabla(P)$. Then, for any $g \in \mathrm{Hom}(P, M)$, $U := \mathrm{Im}\,(fg) \ll M$. However, by Lemma 1.3, $\mathrm{Hom}(P, U) = 0$ and so $fg = 0$. This implies $\mathrm{Im}\,(f) \subseteq \mathrm{Ke}\,(g)$ and so

$$\mathrm{Im}\,(f) \subseteq \bigcap \{\mathrm{Ke}\,(g) : g \in \mathrm{Hom}(P, M)\} = \mathrm{Re}(P, M) = 0,$$

as P is cogenerated by M (see [20, 18.4]). Thus $f = 0$, i.e. $\mathrm{Jac}\,(\mathrm{End}(P)) = 0$.
On the contrary if $\nabla(P) = 0$, then P is copolyform by 2.5. Denote by $p : P \twoheadrightarrow M$ the projection and let $\pi : M \twoheadrightarrow X$ be any small cover. The composition $p\pi : P \twoheadrightarrow X$ is also a small cover and therefore codense. In particular $\mathrm{Hom}(P, \mathrm{Ke}\,(p\pi)) = 0$ and, by projectivity of P, $\mathrm{Hom}(P, \mathrm{Ke}\,(\pi)) = 0$. By 1.3 π is a codense cover, i.e. M is copolyform. □

2.19 The last proposition showed that a projective cover of a copolyform module is copolyform as well.

Corollary *Let M be a copolyform module with projective cover P in $\sigma[M]$, then* $\text{End}(M)$ *is a subring of* $\text{End}(P)$ *such that every epimorphism $f \in \text{End}(M)$ with small kernel is invertible in* $\text{End}(P)$.

Proof Denote by $p : P \to M$ the projection and take any non-zero $f \in \text{End}(M)$. Then by the projectivity of P, there exists a non-zero $\bar{f} \in \text{End}(P)$ such that $pf = \bar{f}p$. Suppose there exists another $g \in \text{End}(P)$ such that $pf = gp$, then

$$0 = pf - pf = (\bar{f} - g)p$$

implies $\text{Im}\,(\bar{f} - g) \subseteq \text{Ke}\,(p)$, i.e. $\bar{f} - g \in \nabla(P) = 0$. Hence $\bar{f} = g$. Thus the correspondence $f \mapsto \bar{f}$ is uniquely defined.

Now assume that f is an epimorphism with small kernel, then $pf = \bar{f}p$ implies that $\bar{f}p$, and hence $\bar{f}$ is an epimorphism with small kernel. By the projectivity of P, $\bar{f}$ splits and, as $\text{Ke}\,(\bar{f}) \ll P$, must be an isomorphism. □

2.20 The existence of a projective cover, turns the class $\mathcal{S}$ of M-small modules into a cotorsion class:

Proposition *Assume that M is dual non-singular in $\sigma[M]$ and has a projective cover P in $\sigma[M]$. Then the class of small modules in $\sigma[M]$ is closed under submodules, factor modules, extensions and direct products (in $\sigma[M]$) and can be described as:*

$$\mathcal{S} = \{X \in \sigma[M] : \text{Hom}(P, X) = 0\}$$

Moreover for any $Z \in \sigma[M]$, $\rho(Z) = \text{Re}(Z, \mathcal{S})$ is dual non-M-singular and $Z/\rho(Z)$ is M-small.

Proof Note that if $\mathcal{S}$ can be described as stated above, then it also satisfies the closure properties. Hence we only need to show that $\mathcal{S}$ equals the class of modules X with $\text{Hom}(P, X) = 0$. Let X be any module in $\sigma[M]$ and $\widehat{X}$ its injective hull in $\sigma[M]$. By [20, 17.9], $\widehat{X}$ is M-generated and hence P-generated. If X is not M-small, then it is not small in its M-injective hull $\widehat{X}$. Thus assume there is a proper submodule Y of $\widehat{X}$ such that $X + Y = \widehat{X}$. Then $X/(X \cap Y) \simeq \widehat{X}/Y$ is a nonzero P-generated R-module. Hence there is an index set Λ and an epimorphism $f : P^{(\Lambda)} \to X/(X \cap Y)$ and so, since $P^{(\Lambda)}$ is projective in $\sigma[M]$, f can be lifted to a homomorphism $g : P^{(\Lambda)} \to X$, i.e. $\text{Hom}(P, X) \neq 0$. Hence $X \notin \mathcal{S} \Rightarrow \text{Hom}(P, X) \neq 0$.

On the other hand assume $0 \neq X \in \mathcal{S}$ and $f \in \text{Hom}(P, X)$. Denote by $Y = \text{Im}\,(f)$ and let $\pi : P \to M$ be the projection. Then extend f to a homomorphism

$$g : M \simeq P/\text{Ke}\,(\pi) \to Y/(\text{Ke}\,(\pi))f$$

sending $p + \text{Ke}\,(\pi)$ to $(p)f + (\text{Ke}\,(\pi))f$. Since M is dual non-M-singular, $g = 0$ and $Y = \text{Im}\,(f) \subseteq (\text{Ke}\,(\pi))f$. Thus $P = \text{Ke}\,(\pi) + \text{Ke}\,(f)$, but since $\text{Ke}\,(\pi) \ll P$, $\text{Ke}\,(f) = P$ and $f = 0$. This shows that $X \in \mathcal{S}$ implies $\text{Hom}(P, X) = 0$ proving the equality of the classes indicated.

Thus $\mathcal{S}$ is closed under submodules, factor modules, direct products and extensions. Note that it follows also that P is dual non-M-singular. Moreover since

$Z/\rho(Z)$ is a subdirect product of M-small modules, it is M-small. Furthermore, since P is projective and

$$\mathrm{Hom}(P, \rho(Z)/\mathrm{Tr}(P, \rho(Z))) = 0,$$

we must have $\rho(Z) = \mathrm{Tr}(P, \rho(Z))$, i.e. $\rho(Z)$ is P-generated and therefore dual non-M-singular. □

In the case above, P generates the *cotorsion theory* whose cotorsion modules are the M-small modules in $\sigma[M]$. the cotorsion free modules are precisely the P-generated modules.

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